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AN INTRODUCTION
TO THE
ELEMENTS OF EUCLID

BEING A FAMILIAR EXPLANATION
OF THE
FIRST TWELVE PROPOSITIONS OF THE FIRST BOOK.

BY THE
REV. STEPHEN HAWTREY, A.M.
LATE ASSISTANT MASTER AT ETON.

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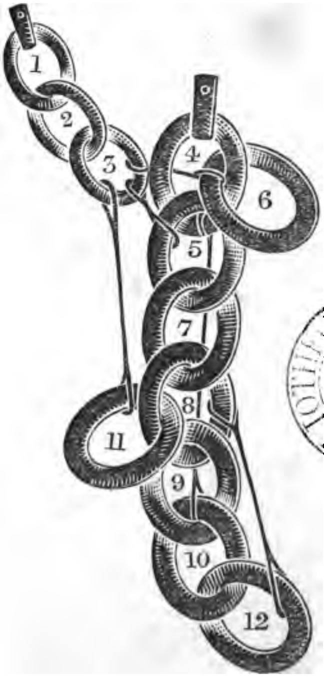

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ΜΥΣΤΑΙΣ

suis

ἀγαπητοῖς

HOC OPUSCULUM DICAT

S. H.

Extract from 'Narrative Essay on a Liberal Education,' by the writer of this Treatise.—'And not only is the Euclid lesson an interest full of charm; it forms a bond of firm friendship in after-life. How often, amidst the mountains and valleys of Switzerland, do I find my hand grasped, and, looking up, I see an animated face, radiant with pleasant memories. "What, don't you know me? I am one of your old μύσται." Alluding to a long and growing list, that used to hang up in the Mathematical School at Eton, of the *Initiated*—that is, of those who understood and could apply the Fourth Proposition of the First Book. These were my μύσται. The Platonic motto "μηδεὶς ἀγεωμέτρητος εἰσίτω" standing at the head of the list shut out the *profanum vulgus*; and hearty—nay, vehement sometimes—were the struggles to get into it.'

PREFACE.

THE FIRST IMPRESSION of any one who takes up this book may possibly be, how much there is here, and, after all, what a little way it goes in EUCLID—only to the Twelfth Proposition.

But teachers know that the first Twelve Propositions of the First Book is the battle-field with Euclid. If the victory is won here, all is easy afterwards. A keen glance and a little sustained thought is all that is needed to master any subsequent Proposition.

And this being so, the learner's progress through Euclid will be as interesting and profitable as it is useless and unprofitable if he has not learned to appreciate geometrical reasoning, and to enjoy its rigour and satisfactoriness.

Perhaps it will form the best preface to what follows if the writer states, not the object of the treatise only, but the circumstances under which it was written.

A happy connection of thirty-six years with Eton has closed, and as the writer spends the first weeks of his retirement among the mountains of Switzerland, many heart-moving thoughts about bygone days arise; especially there comes before him the remembrance of very pleasant hours spent with successive generations of loyal-hearted boys over the early Propositions of Euclid.

He is aware that it is the experience of some teachers that most boys find the study of Euclid repulsive: his experience is the reverse of this; he cannot call to mind the case of a single pupil who found the study of Euclid *repulsive.* They may have found it hard. They may not, in some cases, have been industrious and persevering enough to succeed in examinations. But all of them have felt that there was something real and great in Euclid. If they have failed, they have readily acknowledged that their failure was their own fault, and not because there was no meaning in Euclid.

It would be very heartless, too, if the writer were to forget the many visits full of friendliness, and even affection, which he received from pupils who had passed out of his hands—both during their Eton career, and after they had left Eton—to talk over the old days when they began Euclid together, and to tell him the good which the lessons on the first Twelve Propositions had done them.

Now the writer, in teaching Euclid, always made it his special aim, at the first start, to give life, animation, and suppleness to the cold and rigid form in which Euclid is presented to beginners in their school books. This he found to be most successfully done by familiar talk, appeals to common sense, homely explanations, reference to things of every-day life, and even humorous illustrations, and stories in point.

In the following treatise the substance of such conversations is embodied, with the view of making Euclid a book of life and meaning to those learners who, beginning the study of it in the common, regular editions of the work—in which the style is formal, rigid, accurate—find it *but a book of words.*

What Euclid examiner has not had to mourn over pages of utter trash, mere verbiage, shown up by boy after boy—a feeble imitation of Euclid's phraseology, but giving no sign whatever of their having any insight into the meaning of the words which they were writing.

Now the aim of this treatise is simply to shake learners out of the miserable habit of resting in the words of Euclid; and to lead them to be dissatisfied till they feel that they are taking in his meaning.

Those beginners who do not need the book need not use it. Let them take in hand any of the good modern editions of Euclid—Todhunter's, Pott's, Isbister's, or the Scottish School Book—and grapple with them unaided.

If others find the explanation too profuse and endless; if they find themselves growing weary of the reiteration, let them deal with the book as they do with their corks or swimming-belt. When they want them no longer, they throw them away, and swim alone.

It is the writer's earnest desire to guide and lead on his readers so patiently and helpfully that none shall be left behind. It is the slowest and weakest that he especially wishes to take by the hand. The very difficulty they have in taking in Euclid's meaning shows that they most need Euclid.

Such is the writer's object, and his first thought, in putting into print the substance of his conversational method of teaching Euclid, has had reference to his old Eton pupils. He wishes, though no longer able to speak to them face to face, thus to offer to them the helping-hand that they used to take a friendly grip of.

But he looks to others also with whom he has had no personal connection. The many hearty and very friendly letters which he received from teachers both in England and America when, some years ago, he gave in a tract[1] a sketch of this mode of teaching Euclid, leads him to hope that they may find this treatise of use in their schools.

He trusts, too, that those who are struggling with Euclid, without any one at hand to explain it, and to remove difficulties, may find the treatise a help to them. And he will be glad if some, who look askance, with a half-smile and half-shudder at the idea of reading Euclid, shall be led on by these familiar explanations to a study that will be of incalculable benefit to them.

Another set of readers whom the writer has in his mind are those classes of women who, having tried this or that fragment of a scientific subject, have come to the sound conclusion that they must 'begin at the beginning, and lay a foundation,' and who have, most wisely, chosen Euclid for that foundation.

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[1] *A Narrative Essay on a Liberal Education*, Hamilton & Adams.

To these classes he hopes that his treatise may be useful. But as they will be in earnest, and assiduous, a half-apology is due to them for the homely and humorous illustrations introduced to arrest attention and awaken interest. Let them kindly remember that the following explanations were given to, and are written for, boys, who, high-spirited and thoughtless, are not fully alive to the importance of what they are learning, or bent, as they are, on self-improvement.

It has been already said that these explanations are not written for quick and able boys, but for the great mass of those of average intellect, who have not naturally the accuracy of thought, and the powers of reasoning with which some few are gifted. Yet even an able boy may start at an advantage by a rapid perusal of these familiar explanations. Possibly he may gain a fuller view of Euclid's reasoning than he would have had without them, and pick up thoughts that he might otherwise have missed.

One word more: a body of men, able mathematicians, are now working together with a view of improving our methods of geometrical teaching. The writer deprecates the charge of antagonism with them.

They are to a great degree entering on that profession which he is leaving. Euclid was king in the writer's time, was still seated on the throne which he has occupied for more than two thousand years.

If the common consent of leading mathematicians, in the next generation, decide that he must be dethroned, he must go. But geometrical reasoning will not go. And as the aim of this treatise is to make learners appreciate geometrical reasoning, the writer counts on the sympathy of all who are desirous of improving our geometrical teaching, and are working towards that end.

A WORD of ADVICE to the LEARNER.

PROVIDE YOURSELF with a pair of compasses having a pen and pencil leg, a small flat ruler, and a hard pencil (and keep its point fine). Also it would be well to have by you a little box-wood triangle, called by instrument makers a 'set square.' In default of this, double down and cut off the corner of a sheet of cardboard; it will answer the same purpose.

While you are reading this treatise, have your pen much in your hand, and often try if you understand, and remember what you have read, by writing it out, not copying it.

Remember Bacon's apophthegm:—

Much reading makes a full man,
Much speaking makes a ready man,
Much writing makes a sound man.

INTRODUCTION

TO THE

ELEMENTS OF EUCLID.

EUCLID was born 300 years B.C. He lived at Alexandria, and wrote or compiled, in fact formed into one uniform whole, the treatise on geometry, known as *The Elements of Euclid.*

The work consists of thirteen books. The first six, with twenty-one propositions of the eleventh, and two from the twelfth, form the Euclid commonly read in English schools.

Euclid is looked upon as the father of geometrical reasoning, and so identified is his name with what he wrote about, that the word 'Euclid' in English schools is almost synonymous with 'geometry.'

The first six books of Euclid form a treatise on plane geometry. That is, all the lines you see on opening the pages of any Euclid are supposed to be drawn on a plane or flat surface like the face of the table you write upon. The eleventh and twelfth books are about solid geometry, and refer to lines which might be drawn not only on the face of the table before you, but in any direction either towards the ground or the ceiling or the walls of the room.

On Points, Lines, and Surfaces.

At present you have nothing to do with solid geometry, and yet it may help you the better to understand the definitions which Euclid gives of certain words which he is about to use, if we begin by considering a solid.

Every object which can be held in the hand has three dimensions, commonly called length, breadth, and thickness; and that which has length, breadth, and thickness, is called a solid. It matters not whether the object has a regular form like a brick, or an irregular form like a chance stone that might be picked up.

Let us now take in hand and consider a solid. Here is a brick (see fig. 1), we will say a wooden brick. Its length measured from A to B is so many inches, that is one dimension. Its breadth measured from A to C is so many inches, that is a second dimension. Its thickness measured from A to D is so many inches, that is a third dimension.

Suppose now that its thickness be taken gradually away (planed away for instance) then the brick will become thinner and thinner; and when the thickness has been all taken away, there will remain only two dimensions, length and breadth (that is, in fact, the *face* of the brick). This is Euclid's superficies or surface.

After this, if the breadth of this surface be gradually taken away, it will become narrower and narrower, and when the breadth is all gone, one dimension only remains, that of length (the edge, in fact, of the brick). This is what Euclid calls a line. It has no thickness, no breadth, only length.

Lastly, suppose this line to shrivel up from one end; in other words suppose its length to be gradually taken away, the line will become shorter and shorter, till, when all the length is gone, there will remain no dimension at all; only a dot, as it were, standing at the corner of the brick. Which dot has neither thickness, nor breadth, nor length. All the dimensions of the solid are gone; one after another they have been taken away. Now this dot without length, breadth, or thickness, is Euclid's point; it is called by him *σημεῖον*. And we are now prepared for his

> *Definition* 1.—A point is that which has no parts, or which has no magnitude.

Some commentators on Euclid add,—but position.

Observe, *we* began with a solid brick, and taking it to pieces, resolving it,[1] that is, taking away successively its length, breadth, and thickness, we came at last to a *Point* Now Euclid takes the opposite course: He sets out with a point, and he seems to have before him the idea of the point moving. And as a pencil point moving along paper and leaving a trace behind, forms or generates a visible line, so his point, having no dimensions, no parts, no magnitude, by moving along, will form or generate Euclid's line, which he thus describes in

> *Definition* 2.—A line is length without breadth.

He adds, what is suggested by the idea of the motion of a point generating a line,

> *Definition* 3.—The extremities of a line are points.

He might have added, the intersection of two lines (that is, where they cross one another) is a point.

Now the motion of the point may be curved or crooked. The point may, in its motion, swerve first on one side and then on another, still the moving point (having itself no parts, no magnitude) will generate a line, that is, length without breadth. But if, in its motion, the point do not swerve one way or another, he calls such a line a straight line, for he gives as

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[1] Resolve, ἀνα-λύω, hence analysis.

> *Definition* 4.—A straight line is that which lies evenly between its extreme points.

This much Euclid says of lines, and then he proceeds to the next magnitude of which he speaks; namely, that which has length and breadth.

Here he seems to have before him the idea of the line, which has length only and no breadth, being drawn along in a direction different to its length (as the end of a rake is drawn by its handle), and so tracing or generating a surface which he thus defines:

> *Definition* 5.—A superficies (or surface) has only length and breadth.

He adds, what is suggested by this idea of a surface being generated by the motion of a line in a direction different to its length,

> *Definition* 6.—The extremities of a superficies (or surface) are lines.

In the seventh definition there is given a neat and satisfactory test of a flat surface. It is one naturally used by the polishers of marble surfaces. They take what they call a straight edge, and lay it in different directions on the surface which they are polishing. If the straight edge, wherever it is placed, exactly lies along the polished surface, so that no daylight is visible between the straight edge and the polished surface, then they know that the surface is truly flat. The actual words of the definition or test of a flat surface are these:

> *Definition* 7.—A plane superficies (in other words, a flat surface) is that in which any two points being taken, the straight line between them lies wholly in that superficies (or surface).

Of course if a surface were pushed in any direction except that of its length or breadth, it would generate a solid; in other words, the space through which the surface was pushed would have the three dimensions of length, breadth, and thickness.

Thus beginning with a point, by the method of putting together,[1] that is, by adding successively length, breadth, and thickness we get to a solid. Euclid, however, stops at surface, because the present portion of his treatise is on *plane* geometry.

We may now, then, suppose that we have a flat surface, defined as above, before us, on which to place points and to draw straight, or other lines.

We must here observe that the points which we place on the flat surface must have some magnitude, and the lines which we draw on it must have some thickness, else we should not see them. But it is not difficult to conceive the point to become finer and finer till at last it has lost all its magnitude. In like manner, we may conceive the line to become thinner and thinner, till it has lost all its thickness, and has only the dimension of length remaining. Thus the physical points and lines, for so they are called when they have magnitude and thickness—become mathematical points and lines according to Euclid's Definitions.

On Angles.

Take up your compasses, and open them a little way. The amount of opening between the legs of the compasses is an angle. As you open the compasses more the angle becomes greater, until the legs are drawn so far asunder that they are in a straight line; then there ceases to be (according to Euclid's definition) an angle between them. His definition of an angle being as follows:

Definition 9.[2]—A plane rectilineal angle is

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[1] Putting together, συν-τίθημι, hence synthesis.

[2] Definition 8 is omitted; it refers to angles contained by curved lines, and is not required at present.

the inclination of two straight lines which meet together, but are not in the same straight line.

The legs of the compasses represent the two straight lines. They meet together at the hinge. When the legs are drawn so far apart that they are exactly in opposite directions, they meet together, but *are* in the same straight line.[1]

The forming an angle by drawing the legs of a pair of compasses asunder teaches incidentally what should be early learned and always remembered—that the magnitude of an angle does not depend on the length of the straight lines containing it, but solely on the opening between them.

You may see as you look about you many examples of angles, the corner of the book before you is an angle, so is the corner of the table, or of the door, so again is the corner of a field where two hedge-rows meet. You may remember what Horace calls that cozy corner of his Tiburtine farm, where he hoped to sit in his old age, and drink his Falernian—

ANGULUS *ille mihi ridet.*

The compasses used by smiths and carpenters are sometimes made in the following manner (see fig. 2). From one of the legs a curved rim projects, which moves freely through an opening in the other leg. In this leg there is a thumb-screw, by which the circular rim may be clamped. The legs may be thus kept at any desired distance from each other.

This form of compasses suggests a method of indicating an angle, very commonly used in books on trigonometry, namely, that of putting a dotted curved line between the straight lines which contain the angle. A letter (generally a Greek one, as α), is often inserted in the opening (see fig. 3); and the angle between the straight lines is called the angle α.

---

[1] Hence the following definition of an angle has been suggested: An angle is the difference of direction of two straight lines which meet.

But the learner must not be misled by this figure, to imagine that the angle is the shut-up space, within which the letter α is placed. The dotted line is only put there to draw attention to the amount of opening between the straight lines.

When only two straight lines meet at a point, Euclid indicates the angle between them by a letter (as A) placed at the point of meeting of the two straight lines (see fig. 4), and the angle between the straight lines he calls the angle A.

If more than two straight lines should meet at a point (see fig. $5_{(1)}$), the single letter A would fail to tell whether it was intended to indicate the angle α or the angle β, or indeed the larger angle γ, made up of the angles α and β together.

In this case Euclid uses three letters to indicate the angle. The angle α, that is the opening between the straight lines B A, A C, he indicates by the letters B A C or C A B. The angle β, that is the opening between the straight lines C A, A D, he indicates by the letters C A D, or D A C; and the angle γ, that is the opening between the straight lines B A, A D, he indicates by the letters B A D, or D A B. The letter at the angular point being always the middle letter of the three.

This latter figure (fig. 5) will help the learner to understand the adding angles together, or the subtracting one angle from another. The angle α, which is indicated by the letters B A C, may be added to the angle β, indicated by the letters C A D, and taken together they make up the angle γ, indicated by the letters B A D.

So the angle β (that is C A D) may be taken from the angle γ (that is B A D), and the remainder will be the angle α (that is B A C).

It may be well here to put the learner on his guard against an error, a very odd and unexpected one, sometimes made. When a teacher says to a learner :—‘If from the angle B A D (fig. $5_{(3)}$) you take the angle C A D, what angle remains ?’ it is not uncommon for him to reply, ‘If you take away the angle C A D, all that remains is the line A B.’

Now this is an entire misapprehension. Angles and lines are different things altogether. If an angle is taken from an angle, it is *not* a line that remains, but an angle, just as when you take shillings from shillings it must be shillings, and not inches or ounces, that are left.

In fact, the intermediate line C A (fig. 5) belongs, so to speak, to both angles, when taken with A B, it makes the angle C A B (or α), and when taken with A C it makes the angle C A D (or β). And when either of the two, as α, is taken from the whole angle γ, it is the other angle β that remains. In other words; if from the angle B A D we take the angle B A C, the angle C A D remains.

As it is of the highest importance to see clearly and quickly the angle which is contained between any two straight lines which meet, it would be well for the learner to draw some interlacing straight lines (fig. 6), and to examine himself, or to get some friend to examine him in pointing out the angle contained by any two named lines which meet, or, the angle being pointed out, in naming the straight lines which contain it.

For example. If the friend says, ‘What is the angle contained by L B, B M?’ let the learner place a little curved line between these straight lines, as in fig. 6, to mark the angle; or if the friend puts the curved line between any two straight lines which meet, let the learner name to him the two straight lines which contain the angle so marked.

In naming the straight lines containing the angle pointed out, let the learner be advised to take the first line named *towards* the angular point, and the second *from* the angular point. For instance, if he wishes to name the straight lines containing the angle marked α (fig. 6), let him say that the angle is contained by the straight lines

B A, A D, in preference to saying that it is contained by the straight lines A B, A D, or by B A, D A. This may be thought an immaterial remark, but it is found that learners who name the lines in the way here recommended, learn most quickly to distinguish the angle contained between two named straight lines.

After one more remark in reference to angles, we will pass to the next definition. It will be seen (fig. 6) that the angle contained by B A, A D, by B A, A C, or indeed by M A, A N, is one and the same angle—the one marked in the figure as the angle *a*. This teaches that it matters not at what points in the lines containing an angle we place the letters which indicate it.

On Right Angles, &c.

Suppose A B (fig. $7_{(1)}$), to represent the edge of your desk or table, and C D your ruler. If you bring it to the edge, A B, and hold it there, on end, in such a way that the angles C D A and C D B [these are called adjacent angles] are equal, each of the two is a right angle. If you hold the ruler so as to make one of the adjacent angles greater than the other (fig. $7_{(2)}$), then the greater angle, C D A, is called an obtuse angle (from *obtusus*, blunt), and the lesser, C D B, is called an acute angle (from *acutus*, sharp). This illustration leads on to the Definitions 10, 11, and 12 of Euclid which are as follows:

> *Definition* 10.—When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.
>
> *Definition* 11.—An obtuse angle is that which is greater than a right angle.

> *Definition* 12.—An acute angle is that which is less than a right angle.

On the Circle.

Euclid now passes on to a new kind of magnitude, which is introduced by the two following definitions:

> *Definition* 13.—A term or boundary is the extremity of anything.
>
> *Definition* 14.—A figure is that which is enclosed by one or more boundaries.

The 'figure' so introduced is the circle, which is thus defined:

> *Definition* 15.—A circle (fig. 8) is a plane [or flat] figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within it to the circumference are equal to one another.

He adds:

> *Definition* 16.—And this point is called the centre of the circle.

A primitive mode then of describing a circle would be to fix a pin through your paper into a drawing board, and attaching the point of a pencil to the pin by a thread, to draw the pencil point round, keeping the thread always tight.

A much more convenient method, however, is by means of a pair of compasses, by which the distance of the circumference from the centre can be regulated at pleasure.

The definition of a circle should be remembered and repeated with precision. Frequently in repeating the definition the word 'straight' is left out, also it is a not uncommon mistake to omit the words 'to the circumference,' both which omissions spoil the precision and accuracy of the definition.

Euclid adds three more definitions about the circle. First he defines a diameter thus:

> *Definition* 17.—The diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

To this might be added: A radius is a straight line drawn from the centre of a circle to the circumference. The two next definitions, the 18th and 19th, may be passed over; they are not required for the first twelve propositions.

On Rectilineal Figures, especially Triangles.

With regard to rectilineal figures, Euclid says:

> *Definition* 20.—Rectilineal figures are those which are contained by straight lines.

Such rectilineal figures he divides into three classes, thus:

> *Definition* 21.—Trilateral figures or triangles are those which are contained by three straight lines.
>
> *Definition* 22.—Quadrilateral figures are those which are contained by four straight lines.
>
> *Definition* 23.—Multilateral figures (from *multum*, many; *latus*, side) or polygons (from πολύς, many; γωνία, angle) are those which are contained by more than four straight lines.

In the next three definitions Euclid divides triangles into three classes *according to their sides.*

> *Definition* 24.—If the three sides are equal the triangle is equilateral (from *æquus*, equal; *latus*, side) (fig. 9).

Definition 25.—If two sides only are equal, the triangle is isosceles (from ἴσος, equal; σκέλος, leg) (fig. 10).

Definition 26.—If all the three sides are unequal, the triangle is scalene (from σκαληνός, unequal) (fig. 11).

He then divides triangles into three classes *according to their angles*, thus:

Definition 27.—If one of the angles is a right angle, the triangle is a right-angled triangle (fig. 12).

Definition 28.—If one of the angles is an obtuse angle, the triangle is an obtuse-angled triangle (fig. 13).

[It will be seen hereafter that a triangle can only have one right angle, or one obtuse angle.]

Definition 29.—If all the angles are acute, the triangle is an acute-angled triangle (fig. 14).

Here Euclid closes his definitions about triangles; but as trilateral figures, or triangles, will meet us at every step of our journey through Euclid, it may be well here to make two or three observations about them which may be helpful to beginners.

Obs. 1.—If you indicate an *angle* by using three letters you must be careful, as was before said, to put the letter standing at the angular point between the other two. But if you indicate a *triangle* by using three letters, you may take the letters in any order you like. Thus, the triangle represented in fig. 15 may be called the triangle A B C, or the triangle C B A, or C A B, &c.

Obs. 2.—When you have a triangle before you as A B C (fig. 15), you may consider any one of the three straight lines which form it to be the base on which it stands, and then the two remaining lines are called its *two sides*. Thus if B C be considered the base, the *two sides* are B A, A C. The *vertex* is the point A, where the two sides meet. The angle B A C is *the angle at the vertex*. This angle may also be distinguished as *the angle opposite to the base*, or as *the angle contained by the two sides*. The remaining angles A B C, A C B, are called *the angles at the base*, and it is to be particularly noticed that each of them is opposite to one of the two sides; viz., the angle A B C is opposite to the side A C, and the angle A C B is opposite to the side A B.

As an exercise, it may be well for a learner to take A B, and then A C as the base, and to write down in each of these cases which lines will be the two sides, which point will be the vertex, which will be the angle at the vertex, in other words, the angle contained by the two sides, or the angle opposite to the base; and which will be the angles at the base, or, in other words, the remaining angles; and to what sides they will be opposite.

Obs. 3.—A triangle is a figure bounded by three straight lines; and consequently it has three angles. It also may be considered as a quantity of surface or area enclosed by three straight lines. And when it is said of two triangles that they are equal in every respect, it is meant that the three sides are equal each to each:—that the three angles are equal each to each:—and that the enclosed surfaces or areas are equal. In fact, two triangles equal in every respect, are equal in respect of *sides*, of *angles*, and of *area*.

A learner will find it useful to remember this remark when he is doing the fourth Proposition.

Of four-sided figures the most important is a square, which Euclid thus defines:

> *Definition* 30.—A square is that which has all its sides equal and all its angles right angles (fig. 16).

The remaining definitions of four-sided figures are not essential for our present purpose; we will therefore omit them, as well as all reference to parallel lines.

On the Postulates.

At the close of the Definitions Euclid inserts three Postulates, that is, three things which he asks to be allowed to do (from *postulare*, to ask).

He says, Let it be granted,

Postulate 1.—That a straight line may be drawn from any one point to any other point:

Postulate 2.—That a terminated straight line may be produced to any length in a straight line:

Postulate 3.—That a circle may be described from any centre at any distance from that centre.

In other words, he asks to be allowed the use of a ruler, and a pair of compasses; not to be used for measuring with, but only in order to draw straight lines, to produce (i.e. to lengthen) terminated straight lines, and to describe circles.

It will be enough for a beginner if he takes in this meaning of the postulates, though doubtless they bear a deeper meaning than this.

It is most difficult, or rather impossible, to draw a line mathematically straight. It requires a perfectly straight edge, and a perfectly steady hand, neither of which are attainable. It is still more difficult or equally impossible to produce a terminated straight line, as every draughtsman knows. And to describe a perfect circle, the centre must be a mathematical point, not to speak of other causes of imperfection. Now in the postulates, Euclid asks that it may be granted that his straight lines and circles may be in accordance with his definitions, and ideally correct, that is, free from any imperfection arising either from imperfect instruments or imperfect drawing.

On the Axioms.

We pass on now to the Axioms, that is, to certain propositions or statements which Euclid asks may be taken as true without proof (from ἀξιόω to claim).

Observe: He does not ask that they may be taken as true because they do not *need* proof, but because he has no proof to offer. He proves whatever he can, however self-evident it may be. It is his wish to reduce the number of axioms to the smallest possible.

The first axiom is as follows:

> *Axiom* 1.—Things which are equal to the same thing are equal to one another.

Now this axiom, simple as it sounds, is often, very often, incorrectly applied. The following is an incorrect application of it.

Suppose we have three things, we will call them A, B, and C; and suppose that we know that A is equal to B, and that C is also equal to B. The following is an incorrect application of the axiom. The incorrect conclusion is printed in italics.

We have here A and C, each of them equal to B; and things which are equal to the same thing are equal to one another (Axiom 1).

Therefore A, B, and C are all equal to one another.

This is wrong; the conclusion of the argument ought to have been:

Therefore A is equal to C.

As soon as A has been thus proved equal to C by the first axiom, A, B, and C may be said to be all equal to one another, but not before.

The next six axioms require no comment:

> *Axiom* 2.—If equals be added to equals the wholes are equal.
>
> *Axiom* 3.—If equals be taken from equals the remainders are equal.
>
> *Axiom* 4.—If equals be added to unequals, the wholes are unequal.

Axiom 5.—If equals be taken from unequals the remainders are unequal.

Axiom 6.—Things which are double of the same thing are equal to one another.

Axiom 7.—Things which are halves of the same thing are equal to one another.

The eighth Axiom is as follows:

Axiom 8.—Magnitudes which exactly fill the same space are equal to one another.

This is a very important axiom; a great deal depends upon it. Observe Euclid does not say magnitudes are equal which fill equal spaces, but which fill the same, the very same space.

As an illustration of his meaning, suppose you are casting bullets with a mould. There are two bullets before you which you have cast in the same mould; they are equal in accordance with the eighth axiom, for they have both filled the same space, that is, the hollow of the mould in which they were cast.

In the fourth Proposition great use is made of this axiom. Under certain conditions, there named, Euclid proves that two triangles are equal in every respect (See *Obs.* 3 on triangles, p. 13). Now before he draws the conclusion that the lines, the angles, and the areas required to be proved equal,—*are equal,* you will find that he will be most careful to show that the lines, the angles, and the areas in question may be made, severally, *to fill the same identical spaces.*

We now pass on;

Axiom 9.—The whole is greater than its part;

or, than a part of it.

Axiom 10.—Two straight lines cannot enclose a space.

It takes *at least* three straight lines to inclose *or shut up* a space. If, then, two lines enclose a space, we know that one or both of the lines are not straight for two straight lines *cannot* enclose a space.

Axiom 11.—All right angles are equal to one another.

Modern geometricians have tried to remove this eleventh axiom from the list of axioms, and to give a proof that all right angles are equal; hereafter you may examine such proofs.

The twelfth axiom is not required till the twenty-ninth proposition has been reached, and therefore no allusion to it is required in the present treatise.

On the Propositions.

In every proposition some thing is given as a foundation to start from; for instance, in the first proposition a straight line of a certain length is given; in the fifth proposition a triangle having two equal sides is given. In some of the propositions, starting from the given foundation (or, as it is commonly called, 'hypothesis,' from ὑπο-τίθημι, to lay down as a foundation) you are required to *do* something; for instance, in the first proposition, on the straight line of given length, you are required to describe an equilateral triangle. In other propositions, you are required, from the given hypothesis, to *prove* something; for instance, in the fifth you are required to prove that the angles at the base of the equal-sided (or isosceles) triangle are equal.

The first kind of proposition, in which you have to *do* something, is called a PROBLEM (from προβάλλω, to set before one); the second kind, in which you have to *prove* something, is called a THEOREM (from θεωρέω, to consider). These words are printed in Euclid at the head of each proposition.

It is of the highest importance that a beginner should see clearly in every proposition what it is that is given, and what he has to do or to prove, as the case may be. And no better advice can be given to a learner, who wishes to understand and to profit by Euclid, than this: never to go farther than the heading (or enunciation) of a proposition until he sees clearly what is given, and what has to be done, if it is a problem,—or what has to be proved if it is a theorem. Indeed, so important is it that a learner should see this clearly, that in modern editions of Euclid the enunciations (or headings in which this is stated) are often divided into two paragraphs, the first stating what is given, and the second what has to be done or has to be proved. In the following pages this is done to a greater extent than has been thought necessary in other editions of Euclid.

Observe, further, that each proposition consists of several distinct portions.

First, there is the heading, or general enunciation; that is, the statement of what is given and what has to be done or to be proved, in *general* terms, without reference to any particular lines or figure.

Then follows the particular enunciation. That is, the previous general enunciation is repeated, but now with reference to, and explained by, the lines and figure actually given and drawn.

Then follows, commonly, the construction, in which certain lines, which are necessary in order to do what has to be done, or to prove what has to be proved, are drawn. Sometimes, but very seldom, no construction is required.

Lastly follows the demonstration or proof. In this it is shown (mostly by the help of the lines of construction) that what was to be done *has been done*, or what was to be proved *has been proved.*

The learner is advised to have before him two separate pieces of paper, one to draw the figure on, and the other to write each paragraph, one by one, as soon as its meaning is understood. Great deliberation is strongly recommended on beginning Euclid.

We teachers of Euclid are often much to blame for the quick, off-hand way in which we take pupils through the early propositions. They seem so simple to us, we are so familiar with the ideas and processes of Euclid, that without intending it, we often dazzle, bewilder, confuse our pupils by starting at too quick a pace.

A story which the writer heard in early life from a relative, of the good old days of mail-coaches and posting, has been so useful to himself in teaching Euclid, that he will here repeat it for the benefit of others.

His relative was bound for Ireland; he wished to catch the Holyhead mail. On driving up to the inn-door where the mail changed horses, he found to his dismay that the coach was gone; it had started some minutes. An old wiry post-boy, standing by, said to him, 'I'll catch it.' 'Put your horses to,' was the order. In an incredibly short time the horses are harnessed, and the light, yellow-pannelled post-chaise is brought out of the yard. He gets in and they are off. The impatient traveller was dismayed at finding that the old post-boy started at a deliberate trot, not by any means at the pace of the mail they were to catch, and leaning eagerly forward with his head and shoulders out of the front-window he cried, again and again, 'Drive on, my man, drive on; you'll never catch the mail.' The only answer of the post-boy was a dig back of the right elbow, as much as to say, 'I know what I'm about.'

The traveller, finding the obstinacy of the post-boy immovable, fell back on his seat in despair. But by-and-by he feels that the trot is quickening; at a touch of the post-boy's whip on each shoulder, the horses sprang into a sharp canter, which became a gallop soon; and on they go, now at a full gallop up and down hill, the post-chaise swaying as they sped along. The Holyhead mail is in sight, they dash up along-side; the coachman pulled up for a moment; down came the guard, and while they are hurrying the passenger into the mail, the old post-boy, pocketing his fee, said with a grin, 'I told you I'd catch it. You'd have had me blow my hosses at startin', and then I'd never ha' done it.'

And so he left the passenger to ruminate on the occurrence, and from it to learn a lesson for life. It is here given for the benefit of those, who, whether as teachers or learners, are engaged on the first pages of Euclid.

We will now quietly and deliberately take in hand

PROPOSITION I.

Here our general enunciation is,

Given a straight line:—

It is required to describe an equilateral triangle upon it.

Then follows the particular or explanatory enunciation. The learner should write on one piece of paper:

'1. Let A B be the given straight line' (fig. 17).

And on the other piece of paper he should draw a straight line, and put A at one extremity of it and B at the other. These letters will indicate the extreme points of the line. The extremities of a line are points (Def. 3).

The learner has thus the given line before him; and, keeping his eye on the general enunciation, he can complete the particular enunciation thus:—

'2. It is required to describe an equilateral triangle on A B' (merely putting 'A B' for the word 'it' in the general enunciation).

Then follows the construction:[1]

'1. From the centre A, and at the distance A B, describe the circle B C D.'

The beginner will remember that this, which he is here told to do, is one of the three things which Euclid asks to be allowed to do, namely in Postulate 3.

Let him then do it, placing the steel point of his compass at the extremity A of the line, let him bring the end of its pencil leg to the point B, and sweep round the circle B C D.

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[1] In the course of the construction and demonstration reference is made to the first and third postulates, to the fifteenth definition, and to the first axiom. Some editors strongly recommend that a beginner should repeat, or write out, such references before doing the proposition.

The second step in the construction is:

'2. From the centre B, and at the distance B A, describe the circle A C E.'

Let the learner describe this circle as he did the former; placing the steel point of his compass at the extremity B, and the end of the pencil leg at the extremity A, let him sweep round the circle A C E.

Let the letter C be placed where the two circles cut each other above the straight line A B. The letters D and E may be placed at any other points in the circumferences of the first and second circle.

We now pass to the third step of the construction,

'3. From the point C where the circles cut one another, draw the straight lines C A, C B, to the points A and B.'

This again it will be remembered is one of the three things which Euclid asks to be allowed to do, namely in Postulate 1. Let the learner actually draw the straight lines C A and C B.

The beginner is advised to draw these carefully with ruler and pencil.[1] Place the ruler so that the point of the pencil shall come exactly upon the intersection C of the two circles, and also exactly on the extremity A of the given line. Then draw the line carefully from point to point. In the same careful manner draw the straight line C B. Complete this portion of the proposition by writing the words:

'4. Then A B C shall be an equilateral triangle.'

There remains the demonstration, that the triangle

---

[1] Those who wish to make the figures very neatly should first draw them with a fine pencil, and afterwards ink in the lines with Indian ink, using for this purpose the pen-leg of the compasses, or a drawing-pen. What is *given* should always be 'inked in.

A B C which is here *said* to be equilateral, is really so. Let it be thus written:

'1. A C is equal to A B, because they are both radii of the same circle B C D (Def. 15).'

[If the learner looks back to Definition 15, he will there see that all straight lines drawn from the centre to the circumference of the same circle are equal. In other words, that all radii of the same circle are equal.]

'2. C B is equal to A B because they are both radii of the circle A C E (Def. 15).

'3. Wherefore A C and C B are each of them equal to A B.'

Now look back to Axiom 1, and introduce it into the demonstration by writing thus:

And things which are equal to the same thing are equal to one another.

[Do not write as the conclusion, Therefore C A, A B, and B C, are all equal to one another; that would be wrong (see obs. on Axiom 1, page 15). The right conclusion is,]

'4. Therefore C A is equal to C B.'

Now, indeed, that you have shown

(1) that A C is equal to A B,
(2) that C B is equal to A B,
and (3) that A C is equal to C B:

That is, now that you have shown that each side of the triangle A B C is equal to the other two, you *can* say, Therefore C A, A B, and B C are all equal to one another. Write then as follows:

'5. Therefore C A, A B, and B C are all equal to one another.'

And add in conclusion:

'Wherefore the triangle A B C is equilateral, and it is described on the straight line A B, which was required to be done.'

The First Proposition is thus completed.

As very many learners fall into the mistake of wrongly applying the first axiom in the above and the two following propositions, suppose we try to show how to apply it rightly by a jocular illustration. Horace says

> Ridiculum acri
> Fortius et melius magnas plerumque secat res.

See Figure 18. On the left is drawn a turreted fortress, that is meant to represent the Tower of London; on the right is drawn a castle with a round tower and a flag flying, that represents Windsor Castle. Now the first battalion of the Grenadier Guards is stationed at the Tower, and the second is stationed at Windsor. And you here see two Privates of these battalions pacing backwards and forwards as sentries, one at the Tower, and the other at Windsor Castle.

Now between these two sentries a bet, a heavy and important bet, is pending, as to which of them is the tallest.

Unfortunately, as you perceive, they are both on guard, and cannot therefore decide the question by the simple mode of standing back to back and measuring.

What shall they do? The London man has a friend who measures with him, and finds that his height is exactly that of the sentry.

Now this friend is not on duty, and he gets leave to go to Windsor. In the middle of the picture you see him in the train on his way. When he gets there he measures with the man that is on guard at Windsor Castle, and he finds that his height is exactly that of this sentry also.

Hence the London man and the Windsor man know that they are the same height the one as the other, *though they have not been able to measure the one with the other.*

Remember the important question to be decided is this, whether the London man is equal to the Windsor man. The man that went by the train is only useful as a means of comparing these two.

Look now at figure 17, and say of the three straight lines which form the triangle A B C, which two lines represent the London man and the Windsor man, and which line represents the man that went by the train.

[The learner is advised here to close the book and write the answer to this question, and when he has written it, to read on and see if he is right.]

Ans.—C A and C B are the London man and the Windsor man, and A B is the man that went by the train.

As in the foregoing writing out of the First Proposition, the thread of the reasoning was often broken off by explanatory remarks, before passing on to the next proposition it may be well to write it out again, as it might be written in an examination.

PROPOSITION I. PROBLEM.

General Enunciation.

Given a straight line:—

It is required to describe an equilateral triangle upon it.

Particular Enunciation.

Let A B be a straight line (fig. 17):—

It is required to describe an equilateral triangle upon A B.

Construction.

1. From the centre A and at the distance A B describe the circle B C D (Post. 3).
2. From the centre B and at the distance B A describe the circle A C E (Post. 3).
3. From the point C where these circles cut one another draw the straight lines C A, C B, to the points A and B.
4. Then A B C shall be an equilateral triangle.

Demonstration.

1. A C is equal to A B, because they are radii of the same circle B C D (Def. 15).

2. C B is equal to A B, because they are radii of the same circle A C E (Def. 15).

3. Therefore A C and C B are each of them equal to A B.

4. And things which are equal to the same thing are equal to one another. Therefore A C is equal to C B.

5. Wherefore C A, A B, and B C are all equal to one another, and the triangle A B C is therefore equilateral, and it is described on the given straight line A B, which was to be done.[1]

On the Second Proposition.

We now pass on to the second proposition.

You will find that it depends on (hangs on) the first. Here, then, you have a first instance of what you will observe constantly in your progress through Euclid—the dependence of successive propositions on those which have gone before. As you advance from proposition to proposition, you will find a following one so often linked to and dependent on the next preceding, as to bring to mind the idea of a chain in which the successive links are joined on to, and hang, one from another.

Indeed, it is instructive in going through the propositions of Euclid to hang them, chain-wise, one from the other, linking each to that on which it depends, as is done in the woodcut on the title-page, which will explain itself to you as you proceed. In fact, in the second proposition, you will find that you hang on the second link of your 'chain of reasoning.'[1]

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[1] It will be a good exercise to put the letter F at the other point where the circles intersect, and completing the triangle A B F as above, to prove that the triangle A B F is equilateral, in a similar manner.

The question ever to be asked, what is given, and what has to be done in the Second Proposition, is thus answered in the

General Enunciation.

Given a point, and a straight line:—

It is required from the given point to draw a straight line equal to the given straight line.

Now put a dot (fig. 19) where you like to indicate the place of the given point. You had better put it about the middle of your sheet of paper, and call the point A. Also draw a straight where you like (let one of its extremities be about an inch from A), to indicate the given straight line. Put the letters B and C, one at each end of the straight line, and call it the straight line B C.

This being done, looking back to, or remembering the general enunciation, write as follows the

Particular Enunciation.

'Let A be the given point, and let B C be the given straight line:—'

'It is required from A to draw a straight line equal to B C.'

Before the learner begins the construction, he would do well to listen to a word of advice. It applies to every proposition. Avoid always the miserable habit of copying at once the whole figure as given in the book. The figure ought to be constructed, built up (*construere* means to build) gradually, according to the directions given in the successive steps of the construction.

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[1] The learner is advised, as he finishes each successive proposition, to turn to the title-page, and observe how the corresponding link is hung on the chain. A full explanation of the woodcut is given at the end of the book.

We will now go on to the

Construction.[1]

Write: '(1) From the point A to B, draw the straight line A B.'

Then do this carefully with pencil and ruler, observing the directions given in Prop. I. for doing this. Remember it is done by Postulate 1.

This being done, write:

'2. Upon A B describe an equilateral triangle D A B.'

Now this is to be done by the first proposition. Euclid, however, does not encumber his figure by exhibiting the construction of the previous proposition, on which it depends. But in order to construct the figure neatly and correctly, you should draw lightly in pencil, as in Prop. I., two circles with the centres A and B respectively, and with the radius A B intersecting at the point D. Having thus found D the vertex of the equilateral triangle, the straight lines D A, D B may be drawn to the points A and B, and inked in, and when the ink is dry, the two circles may be effaced, and you have the equilateral triangle as it appears in Euclid's figure of the second proposition.

Another way, in common use among draughtsmen, is to open your compasses so that the distance between the points may be equal to A B. Then with the centres A and B, draw two little arcs, that is, small portions of the circumference, about where you judge the circles will intersect. The intersection of these arcs gives the point D, that is, the vertex of the equilateral triangle, which may be completed as before.

The equilateral triangle being thus described, write down the next step of the construction, which is:

'2. Produce the two sides D A, D B to the points E and F.'

Notice that both the straight lines are produced from the same point D, through A and B. They may be produced as far as you like, to the edge of your paper if you think proper. The producing a straight line is not very easy to do accurately; it requires care. The ruler must be placed very carefully along the straight line to be produced, and before producing it the learner must observe if the point of the pencil, when it is drawn along the ruler, passes accurately from end to end, along the straight line to be produced.

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[1] See note 1, on Prop. 1.

Having produced the straight lines D A, D B to E and F, write:

'3. With centre B, and at the distance B C, describe the circle C G H.'

Then describe it. At the point where this circle cuts the line B F, put the letter G. Also put H at any other point in the circumference.

Then write:

'4. With the centre D, and at the distance D G, describe the circle G K L.' [Observe that D is the vertex of the equilateral triangle, and that the distance D G is determined by the first circle C G H.]

Then do so. Put L at the point where this circle cuts the straight line D E, and K at any other point in the circumference of the circle. And add:—

'5. Then shall A L be equal to B C.'

The demonstration that it is so may be written as follows:

'1. B C is equal to B G, because they are both radii of the same circle, C G H (Def. 15).

2. Also D G is equal to D L, because they are both radii of the circle, G L K (Def. 15).'

[Now observe, D A is a part of the straight line D L, and if D A be taken from D L, the remainder will be A L.

Also D B is a part of D G, and if D B be taken from D G, the remainder will be B G.

We know, too, that D A and D B are equal, because they are the sides of an equilateral triangle.

Also we know that if equals be taken from equals the remainders are equal (Ax. 3).]

We may then continue the demonstration by writing thus:

'3. But D A and D B, parts of them, are equal (construction).

4. Therefore the remainder A L is equal to the remainder B G (Ax. 3).

5. But B G was proved equal to B C.

6. Therefore A L and B C are each of them equal to B G.

7. And things that are equal to the same thing are equal to one another.'

[Do not write, therefore, A L, B G, B C are equal to one another, but write,]

' Therefore, A L is equal to B C (Ax. 1).

Wherefore from the given point A a straight line A L has been drawn equal to B C, which was to be done.'

Having now completed the second proposition, the learner will do well to try if he can detect which two lines are the London and Windsor man, and which line is the man that went by the train. [Having written the answer, let him read on and see if he is right.]

Ans.—A L and C B are the London man and Windsor man respectively, and B G the man that went by the train.

Obs.—It will greatly help you in the demonstration of every proposition to attend to the following remark:—

Whatever you are told to do in the construction is sure to follow the word 'because' in the demonstration. For instance, in the construction of this proposition you were told, with centre B, and at the distance B C, to describe a circle C G H, and in the demonstration you meet the sentence, 'B C is equal to B G because they are both radii of the same circle C G H.'

Again, you were told in the construction, with the centre D, and at the distance D G, to describe the circle G L K;

and in the demonstration you have to say, 'D L is equal to D G, because they are both radii of the same circle G L K.'

Once more: In the construction you were told on A B to describe an equilateral triangle D A B; and in the demonstration you have to say, 'D A is equal to D B, because they are the sides of an equilateral triangle.'

This is an invariable, essential rule in geometrical reasoning, and it is very helpful to a beginner, when he has taken it in.

You will find it an instructive exercise to make, besides the above, three other figures.

The first, by describing the equilateral triangle on the other side of A B (see fig. 20).

The second, by joining A with C, the other extremity of the line B C, and describing the triangle on the upper side of A C (see fig. 21).

The third, by describing the equilateral triangle on the other side of A C (see fig. 22).

It will be a useful exercise, also, to place the given point A somewhere in the given straight line B C. It will not be necessary, in this case, to join A to B, or C, because these points are already joined (A B and A C being parts of the given straight line B C); and from A, as before, a straight line, indeed four straight lines, $A L_1$, $A L_2$, $A L_3$, $A L_4$, may be drawn, each equal to B C (see fig. 23).[1]

Observe the nearer the given point is placed to the end of the given straight line, the smaller becomes the side of the equilateral triangle, and when the given point A coincides with the end of the given straight line, the equilateral triangle becomes reduced to a point, and the two circles become one and the same, the centres of both being the same point.

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[1] This figure looks complicated, because all the four straight lines are drawn in one figure, It might be simplified by drawing four separate figures, as above.

In fact, in this case the whole construction is as follows: With centre B (A and B now coinciding), and at the distance B C, describe a circle; then any straight line, drawn from the centre B to the circumference of this circle, is a solution of the problem. This is the case (you will meet with the like again) in which, under special circumstances, the number of solutions to a problem becomes unlimited; sometimes under special circumstances the solution of a problem becomes impossible.

It remains now, before passing to the third proposition, to write out the second without explanatory remarks, in the way in which it might be written out in an examination.

PROPOSITION II. PROBLEM.

General Enunciation.

Given a point and a straight line:—

It is required from the given point to draw a straight equal to the given straight line.

Particular Enunciation (fig. 19).

Let A be the given point, and B C the given straight line:—

It is required from A to draw a straight line equal to B C.

Construction.

1. From the point A to B draw the straight line A B (Post. 1).

2. On A B describe an equilateral triangle D A B (Prop. 1).

3. Produce the straight lines D A, D B, to E and F.

4. From the centre B, and at the distance B C, describe the circle C G H (Post. 3), cutting D F in the point G.

5. From the centre D, and at the distance D G, describe the circle G K L, cutting D E in the point L.

Then shall A L be equal to B C.

Demonstration.

1. B C is equal to B G, because they are both radii of the same circle C G H (Def. 15).

2. D G is equal to D L, because they are both radii of the circle G L K (Def. 15).

3. D A, D B, parts of them, are equal, because they are sides of an equilateral triangle.

4. Therefore the remainders B G and A L are equal (Ax. 3).

5. But B G was proved equal to B C.

6. Therefore B C and A L are each of them equal to B G.

7. And if two things are equal to the same thing, they are equal to one another.

Therefore A L is equal to B C (Ax. 1).

Wherefore from the given point A, a straight line A L has been drawn equal to the given straight line B C, which was to be done.

On the Third Proposition.

The third proposition is one the point of which is often missed. Its dependence on the second is overlooked.

Here, as in the former propositions, take a fresh piece of paper to describe the figure on—a square piece, of the size of an ordinary copy-book will do; and as there are many lines of construction to be drawn, begin the figure in the middle of the page.

General Enunciation.

To the question, ever to be repeated, What is given in the third proposition, and what has to be done? the answer is, Two straight lines are given, one of which is greater than the other :—

And it is required from the greater, to cut off a part equal to the less.

On a page different from that on which the figure is to be drawn, write thus :

'1. Let A B and C L be two straight lines, of which A B is the greater.'

Then draw these two straight lines ; (let C L be about an inch and a half long, and let the nearest extremities of the straight lines be about an inch apart) ; and add,

'2. It is required from A B the greater, to cut off a part equal to C L, the less.'

The first step in the construction is :

'I. From the point A, draw a straight line equal to C L (Prop. 2).'

This at once links this, the third proposition, on to the second, for what you are here told to do is exactly what you were taught to do in the second proposition.

Regard then at present only the *point* A (which is one extremity of the straight line A B) and the straight line C L. And taking A as the given point, and C L as the given straight line, do all the construction of the second proposition over again.

Obs.—In the fig. 24(1) this construction is given in dotted lines ; do you do it in pencil, that it may afterwards be effaced.

i. The first step of the construction of the second proposition is from the point A to C, to draw the straight line A C (Post. 1).

ii. The second step is to describe an equilateral triangle on A C.

To do this we must go back to the first proposition. But in order not to encumber your figure with circles, find the vertex of the equilateral triangle by the intersection of two small arcs according to the method described on p 27 ;

call the intersection of the arcs H, join H A, and to save time and ensure correctness, *before* removing the ruler, produce H A to any point K. In like manner join H C, and produce it at once to M.

iii. The third step in the construction of the second proposition is this: With the centre C, and at the distance C L, to describe a circle L N R intersecting C M in the point N.

iv. And the last step is, with the centre H, and at the distance H N, to describe the circle N D S intersecting A K in the point D.

By the above construction A D has been drawn from A equal equal to C L.

It may be as well to go over to yourself rapidly the proof given in Prop. 2, that A D is equal to C L. If you are by this time up in the proof, and can follow it with this figure, it will not take a minute. If you are not up in the proof, and are obliged to refer to the demonstration given in the second proposition, it will be good practice to go it over again. And by all means pick out the Windsor man and the London man, and the friend that went by the train. When you have found them out, or think you have, read on and see if you are right.

Ans.—C L and A D are the London and Windsor man respectively, and C N is the man that went by tho train.

And now all the construction indicated by the dotted lines having been done by you in pencil, ink in the line A D, and when the ink is dry, efface all the pencil construction, and you have remaining on your paper the two given straight lines A B and C L, and a third straight line A D drawn from one extremity A of the greater straight line A B, equal to C L, the less (see fig. 24(1)).

And here begins the new part of the third proposition.

There is only one step of construction in this new part; write it thus:

'2. With the centre A, and at the distance A D, describe the circle D E F, cutting A B in the point E.'

You have but to add:—'Then A E shall be equal to C L.'

The demonstration is as follows:

'1. A E is equal to A D, because they are both radii of the same circle D E F.

2. But A D was made equal to C L (Prop. 2).

3. Therefore A E and C L are each equal to A D.

And if two things are equal to the same thing they are equal to one another. Therefore A E is equal to C L (Ax. 1).

That is, from A B the greater of two given straight lines, a part A E has been cut off equal to C L, which was to be done.'

It is hoped that the learner will have no difficulty in detecting that C L and A E are respectively the London man and the Windsor man, and that A D is the man that went by the train.

It only remains now to write out the third proposition, unencumbered with explanatory remarks.

PROPOSITION III. PROBLEM.

General Enunciation.

Given two straight lines, of which one is greater than the other:—

It is required from the greater to cut off a part equal to the less.

Particular Enunciation.

Let A B and C L be two given straight lines (fig. 24(a)), of which A B is the greater:—

It is required from A B, the greater, to cut off a part equal to C L, the less.

Construction.

1. From the point A draw the straight line A D equal to C L (Prop. 2).

2. From the centre A, and at the distance A D, describe the circle D E F, cutting the straight line A B in the point E.

Then shall A E be equal to C L.

Demonstration.

1. A E is equal to A D, because they are both radii of the same circle D E F (Def. 15).

2. But A D was made equal to C L (Prop. 2).

3. Therefore A E and C L are each of them equal to A D.

4. And things which are equal to the same thing are equal to one another; therefore A E is equal to C L (Ax. 1).

Wherefore from the greater of two given straight lines a part has been cut off equal to the less, which was to be done, or Q. E. F. (*Quod erat faciendum*).

On the Fourth Proposition.

We pass on now to the fourth proposition. It is a great and most important proposition, and has been called the key to Euclid. It is in fact the real beginning of Euclid, the three first propositions being introduced before it only because in the fifth and sixth propositions Euclid has from the greater of two straight lines to cut off a part equal to the less.

The fourth proposition is one of a new kind, unlike the first three, which are *problems*, this is, a *theorem*. In the first three something was to be *done*. In the fourth something has to be *proved*.

General Enunciation.

Let us begin with the never-to-be omitted question, What is given in this proposition, and what has to be proved?

The reply is as follows:

Two triangles are given, which have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal:—

And it is required to prove that, in this case, the two triangles will be equal in every respect, namely, that:—

(1) The bases, or third sides, will be equal.

(2) Each pair of the remaining angles, to which the equal sides are opposite, will be equal.

(3) The triangles will be equal; that is, the quantity of surface enclosed between the two sides and the base of one of the triangles, will be equal to the quantity of surface enclosed by the two sides and the base of the other triangle.

It may be useful to look back to page 12 and read over again the observations 2 and 3 on Triangles.

Particular Enunciation (Fig. 25).

Let A B C and D E F be two triangles which have two sides, B A and A C of the one, equal to two sides, E D and D F of the other, each to each, namely A B to D E, and A C to D F. Also let the angle B A C, contained by the two sides B A, A C of the triangle A B C, be equal to the angle E D F contained by the two sides E D, D E of the triangle D E F:—

Before going on to say what has to be proved, it will be well here to pause and give some directions about drawing these two triangles.

First draw in pencil two straight lines, A M and D N (see fig. 25(1)); then take your wooden or cardboard triangle, and lay one of its edges along A M, bringing one of its acute angles to the point A; and using the triangle so placed as a guide and ruler, draw from A a straight line A P; then remove the triangle, and place the same edge of it along the straight line D N, placing on D the angular point which before lay on A, and, in the same manner, from D draw the straight line D Q. Then take your compasses and them mark off from A M and D N equal parts A B and D E, and opening the compasses a little more, mark off from A P and D Q equal parts A C and D F, and join B C and E F. This being done, ink in the sides of triangles A B C and D E F, and effacing the pencil marks denoted in the fig. 25(1) by dotted lines, you have two triangles A B C, D E F (fig. 25(3)), which have two sides of the one equal to two sides of the other, and likewise the included angles equal.[1]

And you have to prove:

First, that the base B C is equal to the base E F.

Secondly, that the remaining angle A B C is equal to the remaining angle D E F (these angles being opposite to the equal sides A C and D F), and that the remaining angle A C B is equal to the remaining angle D F E (these angles being opposite to the equal sides A B and D E).

Thirdly, that the triangle A B C is equal to the triangle D E F; that is, that the space or area enclosed by the three straight lines B A, A C, and C B is equal to the area or space enclosed by the three straight lines E D, D F, and F E.

And now, supposing that you have carefully gone through this long enunciation, have drawn the figure, and that you see distinctly what is given and what you have to prove, it is very possible that your first observation will be as follows: Euclid gives me these two triangles; he tells me the two sides, or legs, of one of the triangles are equal to the two of the other, and that these legs are equally drawn asunder; and he gives me a long proposition to prove that the lines drawn across at the bottom of the legs must be equal. Who would think of denying it? It must be so.

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[1] It was said, in the observations on the postulates, that your compass and ruler were not to be used for measuring. That only refers to lines and angles which have to be made or proved equal, not to those which are *given* equal; when lines and angles are given equal, you may use any mechanical means for making them so.

Anybody in his senses must see that the cross lines at the bottom must be equal.

To this observation the reply is:

What you say is natural, reasonable, and quite true, and (it might be added) shows active intelligence on your part. Yet, nevertheless, if instead of resting satisfied with '*feeling*' it must be so, you will listen to Euclid *patiently*, and first let him tell you the axiom on which he is going to prove the proposition, and follow him as, step by step, he proceeds to prove that the bases, the remaining angles, and the triangles are equal, in accordance with that axiom, and take in all the steps of his reasoning, so as to be able to reproduce them correctly without assistance—if you are able to review the whole proposition, and to feel that you have clearly and rigorously proved the truth of the statements he makes in the enunciation—by this mental process, you will be lifted up into a higher intellectual level than you have yet reached, or could reach, without this mental effort, and from that higher level you never can come down again all your life long.

Do not think the above to be an exaggerated statement of an enthusiastic teacher of Euclid. There is no mathematician who would not agree that what is here said is no more than *sober truth.*

This is said to encourage you to give attention to the reasoning, to follow what is said with thought and *patience*, and not to allow any statement to pass of which you do not understand the meaning. On the other hand, every step of the reasoning is so simple that there is fear that the simplicity of the steps may be a bar to your taking in the reasoning.

There is nothing abstruse in what you are going to do. Every step of the reasoning is as simple and straightforward as it is rigorous.

Having said this, we will now go back to the proposition. And first we have to prove that the base B C is equal to the base E F.

It is done by the eighth axiom, which asserts, 'That magnitudes which exactly fill the same space are equal' (see the remark on this axiom, page 16). We must therefore show that B C and E F may be made exactly to fill the same space.

With this end in view, imagine the triangle ABC to be taken up and laid upon the triangle D E F, A being placed on D, and the straight line A B being laid on D E, as far as it goes.

Obs.—This you can do with any two figures, however unlike they may be, provided only that they be rectilineal. Here take up a book D E F G, and your wooden triangle P Q R. The book and triangle are quite unlike in form, yet you can put one of the angular points P of the triangle on the corner of the book, and lay the triangle on the book so that a side of the triangle P Q shall lie along the edge of the book D E, *as far as it goes* (see fig. 26(a)). This is all that Euclid tells you to do.

It is not uncommon for learners to begin this proposition in some such way as the following: Put A on D, and B on E, and C on F (fig. 25). But this is altogether wrong, for this you could not do with *any* two triangles which you might chance to take up, and what Euclid tells us to do might be done with such chance triangles, or indeed with a triangle and a book, as we have seen.

Returning now to the two triangles A B C and D E F: A is placed on D, and A B, as far as it goes, is made to lie along D E. But how far will it go? where will the end of it, B, come? where will it naturally and of necessity fall? and why? That is now the question.

Do you hesitate to answer? Well, then, as Euclid's argument is simply an appeal to our common sense, to our every-day experience, we will have recourse to a very homely illustration of his reasoning. Here are a couple of walking-sticks; one of them is two inches longer than the other. Now, if the handles are brought together, and one of the sticks is laid along the other, will the points come together? You feel they will not, and the reason why, you will give readily, 'One of them is longer than the other.' That is quite right. But here are two other walking-sticks; these are of equal length. If we put the handles of *these* together, and lay one of the sticks alongside the other, will *their* points come together? You answer, 'Yes, of course they do.' Why? 'Because the sticks are of equal length.' Exactly so; that is all right. Now turn once again to the triangles. Look upon A B and D E as representing the two walking-sticks; consider the extremities A and D as the handles. Now, A has been placed on D, and A B has been laid along D E, as far as it goes; then how far will it go? where will its extremity B come? will it come on E? 'Yes,' you reply, 'if A B is equal to D E.' But if you look at the enunciation you will see that these lines were given equal; so that you *can* say, B will come on E [not because you put it there, but] because A B was given equal to D E. Are you disposed to say, 'Is that all?' It is all; possibly you may feel disposed to add, 'There is nothing in it.' There is a great deal in it. There is a bit of reasoning perfectly simple, perfectly conclusive. All the succeeding steps are equally so.

We will proceed. Take up again the wooden triangle P Q R and the book D E F G fig. 26(1). It is, you may observe, one of the acute angles of the triangle which has been placed on the corner of the book. You have one side of the triangle P Q lying along the side edge of the book, as far as they go together. Will the other side of the triangle P R lie along the top edge of the book? Clearly it does not. But if the triangle is turned round, and the right angle of the triangle is placed on the corner of the book (fig. 26(2)), and if, as before, one side Q R of the triangle is made to lie along the side edge D E of the book, does the other side Q P of the triangle lie along the top edge of the book, as far as they go together? Yes, now it does. Why is it that in this last case the second sides come together, while in the former they did not? Simply because in the last case the included angles are equal, both being right angles, but in the former they were not.

Turn once again to the two triangles A B C and D E F (fig. 25(2)). We placed A on D, and A B along D E; will A C lie along D F? Certainly, because the included angles B A C and D E F are equal. You remember these angles were given equal, and we took pains, in drawing the triangle, to make them so.

We will pause here to introduce a story. There is often a great deal of teaching to be got from a story.

A boy in going through the fourth proposition said thus: 'Because A B is equal to D E, and the angle B A C is equal to E D F, therefore the line A C lies on D F.' 'No, no,' said the teacher. 'What ought he to say?' The next boy answered, 'Because A B *lies on* D E.' A little fellow sitting beside the speaker (he is a post-captain in the navy now) here looked up eagerly into the teacher's face, and said, 'Please, sir, will you tell me why it is not as good to say it one way as the other?' 'Do you wish to know? I will tell you willingly,' was the reply. 'Look at those two triangles on the black board, A B C and D E F. You remember A B and A C are respectively equal to D E and D F, and the included angle B A C is equal to the included angle E D F.' He answered, 'Yes.'

'Now,' said the teacher, 'if I were to take a saw, and cut down through the board between the triangles, and were to send one half of the board, namely, that half on which the triangle A B C is drawn, to Oxford, and the other half, that on which D E F is drawn, to Cambridge, the lines and angles would still be equal, would they not? Of course they would. But if one of the triangles were at Oxford and the other at Cambridge, could the line A C *lie on* the line on D F? Certainly not.

'But if the triangles are neither at Oxford nor at Cambridge, but there on the board before us, and I conceive the triangle A B C to be laid on D E F, laying A on D, and A B along D E, then indeed A C will lie along D F, because the included angles are equal.'

A flush suffused the boy's countenance. 'Thank you, sir,' said he. 'I see it perfectly now; I never saw it before.' —*Narrative Essay on a Liberal Education*, p. 62.

To resume the demonstration. We have now shown that A C will lie on D F, as far as it goes, and given the reason why. The last question is, Will the point C come naturally and of necessity on F? Yes, it will—and why? Because the straight lines A C and D F were given equal.

[It would be paying you a poor compliment to take up the walking-sticks again to illustrate this.]

We will here review what we have done. Supposing the triangle A B C laid on the triangle D E F, as Euclid tells us to lay it; which is as *any triangle* whatever might be laid on *any other*, we have shown that in the case of these *two particular triangles*, in consequence of their having two sides of the one equal to two sides of the other, and the included angles being equal, *B will come on E, and C on F.* A little reflection will tell that if *any one* of these three given conditions were wanting, we could not prove it.. But all three of them being given, we *can* prove rigorously and satisfactorily that B *must* come on E, and C on F.

In the fig. 25(2) a few strokes are drawn across the triangle A B C. This is done to remind the learner that he is not to conceive that the triangle A B C is still in its old place, but that it is now lying on D E F; A lying on D, it having been placed there; and B and C lying on E and F, where it has been proved that they will come. For the same reason the second letters A, B, and C have been placed at the angular points of the triangle D E F.

We now go forward:

Look at, and for the present confine your attention wholly to, the two straight lines B C and E F (do not think about the triangles). Observe that B is the beginning and C is the end of one of these straight lines, and that E is the beginning and F the end of the other of them; and it has been shown that the beginning B lies on the beginning E, and that the end C lies on the end F. Now these two straight lines being *together* at their beginning and at their end, is it possible for them to separate at all between the beginning and the end? 'that is the question.' Here are a couple of pens; if the feather ends are brought together and the quill ends are brought together (see fig. 27), they do separate between the beginning and the end, and by thus separating they enclose or shut up a space; but that is because the pens are not straight. Two *straight* lines cannot enclose a space (Ax. 10); it takes at least three straight lines to enclose or shut up a space. Therefore, the two straight straight lines B C and E F, being together at their beginning and together at their end, must keep together from beginning to end.

What is the end of all this? Why this: if you now draw the point of your pen along the line joining B and C, and again along the line joining E and F, you find that you have in both cases drawn the point of your pen through *the same space.* But magnitudes which fill the same space are equal. Therefore the base B C is equal to the base E F (Ax. 8).[1]

We have thus proved in accordance with the eighth axiom, that the base B C is equal to the base E F. It will be a slight task compared with the foregoing to prove that the remaining angles and that the triangles are equal.

These are proved equal by the same axiom; we must therefore see if they fill the same space. First let us consider the angles A B C and D E F; confine your attention wholly to these angles, they are contained by the straight lines A B, B C, and D E, E F respectively. And where is A B now lying? It is lying along D E, for *we put it there.* And where is B C now lying? It is lying along E F; *we have just proved that it must come there.* If now you make the little curved rim as described page 6 (see fig. 3) to indicate the angle contained by the straight lines A B, B C (fig. 25(3)), that is, the angle A B C ; and again, if you make a similar curved rim to indicate the angle contained by the straight lines D E, E F, that is, the angle D E F, you find that you draw the curved rims in both cases through the *same* space ; that is, the angles A B C and D E F fill the *same* space, and therefore, in accordance with the eighth axiom, they are equal.

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[1] If we show that two lines begin together, end together, and keep together from beginning to end, this is quite enough to prove that they fill the same space; for lines have no thickness; they can only fill or occupy space of one dimension—that is, length.

The angles A C B and D F E (see fig. 25(3)) are proved equal in precisely the same manner. A C is now lying on D F [not because we put it there, as in the case of A B and D E, but] because, having put A B on D E, we proved that A C must lie along D F, and C B, as said above, has been proved to lie along F E, so that the two rims, if drawn to indicate the angles A C B and D F E, would both be drawn through the same space, thereby showing that the two angles A C B and D F E both fill the *same* space, and therefore again by the eighth axiom they are equal.

Once more. In order to prove that the triangles are equal, we follow the same line of reasoning, thus : The straight line A B was placed on D E, and hence we proved that A C must lie along D F, and that B C must lie along E F ; hence evidently if you put your finger first on the space shut up within the three straight lines A B, B C, and C A, and again on the space shut up or enclosed within the three straight lines D E, E F, and F D, you find you are putting your finger, in both cases, in the same space ; that is, the triangles A B C and D E F both fill the same space,[1] and therefore they are equal (Ax. 8).

It is all over now. You have thoroughly and rigorously proved, in accordance with the eighth axiom, that, if two triangles have two sides of the one equal to two sides of the other, and have likewise the angles contained by those sides equal, they are equal in every respect—

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[1] It is only necessary to show that the boundaries (see Def. 13) of the triangles lie together, in order to prove that the triangles fill the same space, for triangles are surfaces which have only length and breadth, and no thickness.

$$\begin{cases}\text{Equal in respect of sides,}\\ \text{Equal in respect of angles,}\\ \text{Equal in respect of area.}\end{cases}$$

It only remains that the proposition be written for your guidance in the form in which it might be written in an examination.

PROPOSITION IV. THEOREM.

General Enunciation.

It is given that two triangles have two sides of the one equal to two sides of the other, and have likewise the angles contained by those sides equal:—

It is required to prove that the triangles are equal in every respect; that is, that their bases are equal, that their remaining angles are equal, each to each, to which the equal sides are opposite, and that the triangles are equal.

Particular Enunciation.

Let A B C and D E F, (fig. 25(3)) be two triangles which have the two sides B A, A C, equal to the two sides E D, D F, each to each, viz. A B to D E, and A C to D F, and have likewise the included angle B A C equal to the included angle E D F:—

It is required to prove that the triangles A B C and D E F are equal in every respect; that is, that the base B C is equal to the base E F, and that the remaining angles are equal, each to each, viz. A B C to D E F, and A C B to D F E, also that the triangles A B C and D E F are equal.

Construction, or Quasi-construction.

Apply the triangle A B C to the triangle D E F, so that the point A may be on the point D, and that the straight line A B may lie along the straight line D E.

Demonstration.

1. Then the point B will come on the point E, because the straight lines A B and D E are equal (hypothesis).

2. And A B lying along D E, A C will lie along D F, because the included angles B A C and E D F are equal (hyp.).

3. And A C lying along D F, the point C will come on the point F, because the straight lines A C and D F are equal (hyp.).

4. And it has been proved that B will come on E.

5. And B lying on E, and C on F, the straight line B C must lie along the straight line E F from beginning to end, for if they separated, between the beginning and the end, they would enclose a space, which is impossible (Ax. 10).

6. Therefore the two straight lines B C and E F fill the same space, and are therefore equal (Ax. 8).

7. Also the angles A B C and D E F fill the same space, and are therefore equal (Ax. 8).

8. Likewise the angles A C B and D F E fill the same space, and therefore are equal (Ax. 8).

9. And the whole triangles A B C and D E F fill the same space, and are therefore equal (Ax. 8).

Wherefore if two triangles have two sides of the one equal to two sides of the other, and have likewise the angles contained by those sides equal, they are equal in every respect. Q.E.D.

N.B. The words of the enunciation 'to which the equal sides are opposite' must not be overlooked. They enable us to say which of the two remaining angles of each triangle form the pair of equal angles. Here A C is given equal to D F. Now the angle opposite to A C is A B C, and the angle opposite to D F is D E F. Hence the angles A B C and D E F, which are opposite the equal sides A C and D F, form the *one* pair of remaining angles which are equal to each other. Similarly the angles A C B and D F E, opposite to the equal sides A B and D E, form the *other* pair of remaining angles which are equal to each other.

Appended Note on the Fourth Proposition of Euclid, addressed to Teachers.

As the writer has departed a good deal from the usual phraseology in the fourth proposition, especially in avoiding the use of the word 'coincide,' it is right that he should state his reasons.

The fourth proposition will have done its great good to the learner's mind, when he has fully taken in the meaning of the eighth axiom, and felt that, in every respect in which the triangles have to be proved equal, the reasoning has shown that the axiom is fulfilled.

In order to help a learner to do this, he thinks that every obstacle to the right understanding of the axiom, and of the reasoning by which its application is shown, should be put aside.

Now the use of the word 'coincide' in the commonly received phraseology of the fourth proposition is (according to the writer's experience) an obstacle to the learner's taking in the meaning of the axiom, and the reasoning by which it is applied, which in *kindness* to him had better be removed.

In the course of the demonstration the word coincide is used in three different senses, besides its own rigorous, technical sense, as given in the eighth axiom.

First, it is used for coincide in position. Secondly, it is used for coincide in direction. Thirdly, it is used to convey the idea of not separating.

It is not till the fourth time of using the word that it is used in Euclid's full, technical sense of exactly filling—occupying wholly—the same space.

The result of this is, that the minds of learners become so confused that they are as ready to say, at the close of the proof, 'are equal and coincide,' as to say 'coincide and are equal;' indeed they very often (almost invariably) leave out the word 'coincide' altogether;—by these mistakes showing that they have wholly failed to take in the meaning of the demonstration.

Teachers will readily bear witness how often the bewildered pupil (when corrected) replies, 'But to coincide means to be equal.' Now, as coincide means no such thing, the writer has thought it better in this treatise to remove this difficulty out of the learner's way by using for the word 'coincide' what it really does mean.

It may be objected that by using, instead of the word coincide, its synonyme, 'exactly fill the same space,' he brings into prominence what is gently disguised by the word coincide; viz., that certain magnitudes (lines and angles), are said to fill a space, when from their very nature they can fill no space.

The writer holds this to be as nothing compared with a right appreciation of the axiom and its application.

It may be true that lines, having length only, and no breadth, can fill no space; yet if it is proved that they begin together, that they end together, and continue together from beginning to end, then it is shown in an intelligible and very real way that the two lines do exactly fill, or wholly occupy, the same space *as far as their nature enables them to do so.*

Again, if it is shown that the two straight lines containing one angle, lie (as far as they go) exactly along the two straight lines which contain another angle, then we show in a very real and intelligible way that the two angles fill the same space *as far as angular magnitude can do so.*

Indeed, although the lines containing the angles to be proved equal do actually, in the fourth proposition, coincide according to Euclid's full, technical meaning of the word, yet the writer thinks it positively desirable, with the view of keeping the learner's ideas clear on this point, to put out of view the technical, full coincidence of the containing lines. It is their coincidence in direction (or according to our familiar phrase 'as far as they go') that is alone needed, in order to show that the angles contained by them exactly fill the same space, in the only way in which the magnitudes under consideration (angular magnitudes) can do so.

So when we come to the coincidence and consequent equality of the triangles, that is of the enclosed areas, the pointed, prominent thought that should be brought before the pupil's mind is, *not* that the straight lines enclosing the space do *per se* fulfil Euclid's full definition of coincidence, but that, *considered as the boundary lines* of the enclosed spaces to be proved equal, they lie one upon and along the other; by this means showing that the enclosed areas (and it is on the enclosed areas that the pupil's thoughts should now be concentrated) fill the same space.

The question whether pupils will get more readily a true insight into the meaning of the fourth proposition by using or not using the word 'coincide' must be left to the experience of teachers, '*solvitur docendo.*'

Exercises on the Fourth Proposition.

Supposing that you have now mastered the fourth proposition, you have next to be told that as you go on through Euclid you will very often have to employ this proposition to prove that triangles occurring in subsequent propositions are equal in every respect.

Henceforth wherever you can detect that two triangles occurring in any proposition have two sides of the one equal to two sides of the other, and have likewise the angles contained by those sides equal, you can say at once: '*Therefore by the fourth proposition the two triangles are equal in every respect;*' for that is what the fourth proposition has proved.

Now the fifth and sixth propositions are nothing but exercises on the fourth proposition. You have to prove what is stated in the enunciations of those propositions by showing that certain triangles occurring in them are equal in every respect by the fourth proposition. But they are hard exercises. Many much easier exercises occur in subsequent propositions. These subsequent propositions, however, could not be introduced before the fifth, because they depend on the fifth, as you found the third proposition to depend on the second.

And although, on this account, in a regular treatise, these subsequent propositions could not be introduced before the fifth, there will be no harm in bringing in here (in this introduction to Euclid) such propositions or fragments of them, if you know that the object of giving them to you now is to help you, and to lead you on to harder exercises, and that they will afterwards come before you, in due course, in their proper places in Euclid. It will, however, be necessary, in this case, that, for the present, you take for granted a few things that you will hereafter prove in Euclid. Such as these:

1. That you know how to draw a straight line cutting an angle into two equal angles; or, at least, that it can be done.

2. That a straight line can be drawn perpendicular to a given straight line.

3. That if two straight lines cross each other like the legs of a scissors, the angle or opening between the blades is always equal to that between the handles (see fig. 28).

4. That a square may be described on a given straight line.

We will now put before you a graduated series of exercises on the fourth proposition.

EXERCISE I. (fig. 29).

Let A B C be a triangle in which the side A B is equal to A C, and taking for granted (for the present) that we can and have drawn the straight line A D so as to divide the whole angle B A C into two equal angles, viz., B A D and D A C [remember, it is the *angle*, not the *triangle*, that you are to suppose divided into two equal parts]:—

It is required to prove that B D is equal to D C.

In this and all the subsequent exercises the following rules are to be attended to:

1. Observe the two triangles you are about to compare, and find out which of the three sides is to be the base of each triangle. In this exercise the two triangles to be compared are A B D and A C D; and there is no doubt here which are the bases, for we are told to prove that B D is equal to D C; and the sides to be *proved equal* are always to be taken as the bases.

2. Having found which is the base, then the two sides and the angle contained by the two sides are immediately known.[1] Refer now to the enunciation, in order to find if any statement is there made which makes you know that one of the sides of one of the triangles is equal to one of the other. On doing so, you find that the side A B is given equal to A C. It is so stated in the enunciation.

Write therefore as follows:

'1. A B is equal to A C, for it was so given or stated in the enunciation; or more briefly, "by the hypothesis," or simply (hyp.).'

Then do the same for the second side. N.B. Sometimes the equality of the sides is to be gathered from the construction. In this exercise, the second side A D is a side of each of the triangles; in other words, is common to both.

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[1] See Observation 2, on triangles, p. 12.

Therefore write next:

'2. A D is common to both triangles.'

Next it is desirable, especially at first, in order to fix in your mind what the two sides are, to repeat Steps 1 and 2, only now taking *together* the two sides of each triangle.

Write therefore thus:

'3. Wherefore the two sides, B A, A D of the triangle B A D are equal to the two sides C A, A D of the triangle C A D.'

Your naming the two sides of each triangle thus, makes you see what the angle contained by the two sides—in other words, the included angle—is; for when you have named that the two sides are B A, A D, you see that the included angle is B A D; and so in the other triangle when you have named that the two sides are C A, A D, you see the included angle is C A D. Look then back to the enunciation, and see if you are told there that the included angle B A D is equal to C A D. You find it is given or supposed equal, and you write thus:

'4. And the included angle B A D is equal to the included angle C A D, because it was given that the whole angle B A C was divided into two equal parts, or bisected, by A D.'

Having thus shown that in the two triangles A B D and A C D two sides of the one are equal to two sides of the other, and that the angles contained by those sides are equal, you conclude the exercise thus:

'5. Therefore by the fourth proposition the two triangles A B D and A C D are equal in *every* respect; wherefore the base B D is equal to the base C D; which was to be proved.'

If in this exercise, instead of being told to prove that the base B D was equal to D C, you were told to prove that A D was at right angles to B C; the exercise would be done exactly as explained above, as far as the words, 'Therefore, by the fourth proposition, the two triangles are equal in every respect.' But instead of closing with the words, 'Wherefore the base B D is equal to the base C D,' you would have to close the exercise thus: 'Wherefore the remaining angles, to which the equal sides are opposite, are equal; that is, the angle A D B is equal to the angle A D C;' you would have to add, 'and these are adjacent angles; but when one straight line falling on another straight line makes the adjacent angles equal to each other, each of them is a right angle (Def. 10). Wherefore A D is at right angles to B C, which was to be proved.'

Observe that although in this form of the exercise you are not told which are the bases, there is no difficulty in finding them out; for the bases are always the sides opposite to the angles *given* equal. Here the angles given equal are B A D and C A D, therefore B D and D C, the sides opposite to them, are the bases.

The above exercise has been so interspersed with explanatory remarks that it may be well to write it out, as it might be shown up for an exercise.

Enunciation.

It is given that A B C (fig. 29) is an isosceles triangle (see Def. 25), and that the angle B A C is bisected by the line A D :—

It is required to prove that the base B D is equal to the base C D.

Construction (none).

Demonstration.

1. A B is equal to A C (hyp.).

2. A D is common to both triangles.

3. Wherefore the two sides B A, A D of the triangle B A D are equal to the two sides C A, A D of the triangle C A D.

4. And the included angle B A D is equal to the included angle C A D (hyp.).

5. Therefore by the fourth proposition the two triangles

B A D and C A D are equal in *every* respect; wherefore the base B D is equal to the base C D, which was to be proved.

In the following exercises a few hints will be given; and it is advised that, with the help of those hints, the learner should try to write out the five steps of the demonstration. The five steps are written out in each case, fully, in this book, in order that the learner may see if he is right. If in any case he finds the exercise too difficult to be done without help, he can refer to the book for assistance; still let him not lose courage, but try to do the next exercise without help.

EXERCISE II. (fig. 30).

Let A B be a straight line, of which C is the middle point. Let it be granted that C E is drawn at right angles to A B from the point C. Let D be any point in E C, and let A D and B D be joined:—

It is required to prove that A D is equal to B D.

Hints.

1. Here the two triangles to be compared are clearly A C D and B C D, and A D, D B are the bases of the triangles, for these are the sides to be proved equal.
2. These being the bases, the two sides of the triangle A C D are A C, C D, and the two sides of the triangle B C D are B C, C D, and the angles contained by these sides are A C D and B C D.
3. Now of these sides, A C and C B are given equal (see enunciation); also C D is a side of both triangles; and the included angle A C D is equal to the included angle B C D (see Ax. 11).

With these hints let the learner try to write out the five steps of the demonstration without, for the present, reading further. When he has written them out, he can read on and see if he is right.

Demonstration.

1. A C is equal to B C (hyp.).
2. C D is common to both triangles.
3. Therefore the two sides A C, C D of the triangle A C D are equal to the two sides B C, C D of the triangle B C D, each to each.
4. And the included angle A C D is equal to the included angle B C D, because they are right angles (Ax. 11).
5. Therefore by the fourth proposition the two triangles A C D and B C D are equal in *every* respect; wherefore the base A D is equal to the base B D, which was to be proved.

EXERCISE III. (fig. 31).

Let A B C be a triangle. Let it be granted that E is the middle point of A C, join B E, and produce B E to F, making E F equal to B E (this can be done by Prop. 3), and join F C :—

It is required to prove that the angle B A E is equal to the angle E C F, and that the angles A B E and E F C are also equal.

In the demonstration of this exercise you may take for granted that when a pair of scissors are opened, the angle between the blades is equal to the angle between the handles ; in other words, that when two straight lines as A C and B F cross each other or intersect, the vertical or opposite angles A E B and C E F are equal. This is proved in the fifteenth proposition.

Hints for the Solution of Exercise III.

We have first to find out which are the two triangles to be compared. Clearly they are the triangles A E B and C E F, for the angles to be proved equal are angles of these triangles.

Secondly, we have to find out which of the sides of these triangles are in this exercise to be taken as the bases.

It may be well here to enumerate the different rules which may be given for discovering the bases, and to show that they all point to the same lines.

Rule I.—We know that in the fourth proposition the angles to be *proved* equal are the angles at the base; that is, angles of which the base is one side; but here the two angles E A B and E B A (of each of which B A is a side) have to be proved equal to the two angles E C F and E F C (of each of which C F is a side). This rule, therefore, points out A B and C F as the bases of the triangles to be compared.

Rule II.—The bases are the sides opposite to the angles which are given equal. In this exercise the angles given equal are the vertical and opposite angles A E B and C E F. This rule also points to A B and C F, which are the sides opposite these angles, as the bases.

Rule III.—Of the three sides of one triangle that which is *not* given equal to a side of the other, is the base. In this exercise A E is given equal to E C, and E F is made equal to B E. This rule also points to A B and C F (the sides not given equal) as the bases.

Now, knowing which sides are to be taken as the *bases*, we therefore know which are the two *sides* in each triangle, and which are the included angles.

With the above hints, let the learner try to write out the five steps of the demonstration, without, as yet, reading further.

Demonstration.

1. The side A E is equal to the side E C (hyp.).

2. The side B E is equal to the side E F (construction). [They were made so by Prop. 3.]

3. Wherefore the two sides A E, E B of the triangle A E B are equal to the two sides C E, E F of the triangle C E F, each to each.

4. And the included angle A E B is equal to the included angle C E F, because they are vertical and opposite angles [which in this demonstration we are allowed to consider equal].

5. Therefore by the fourth proposition the two triangles A E B and C E F are equal in every respect. Wherefore the remaining angles *to which the equal sides are opposite* are equal; viz., the angle B A F is equal to the angle F C E, and the angle A B E to the angle C F E; which was to be proved.

Exercise IV. (fig. 32).

In this exercise it is allowed, or assumed, that a square may be described on a given straight line. The way to do this is shown in the forty-sixth proposition.

Enunciation.

A B C is any triangle. It is assumed that on the two sides of it, A B and B C, squares are described; viz., A E D B and B F G C.

Also D C and A F are joined:—

It is required to prove that D C and F A are equal.

Hints.

D C and A F, being the straight lines to be proved equal, are therefore the bases of the triangles to be compared. Now the triangles of which D C and A F are bases are D B C and A B F. These, then, are the triangles to be compared. And as we know that D C and A F are the bases, we know at once that D B, B C are the two sides of the triangle D B C, and that A B, B F are the two sides of the triangle A B F.

We must now examine if these sides and included angles are known to be equal.

1. Is D B equal to A B? Yes, they are two sides of a square (see Def. 30).

2. Is B C equal to B F? Yes, for the same reason.

3. Is the angle D B C equal to the angle A B F? This is not evident at first sight, but a little reflection will show that it is. For the angle D B A is equal to F B C (see Def. 30 and Ax. 11); and if to each of these equal angles the

angle A B C be added, we know (by Ax. 2) that the whole angle D B C is equal to the whole angle A B F.

With these hints, now try to write out the exercise without reading further at present.

Demonstration.

1. The side A B is equal to the side D B, because they are two sides of the same square.

2. The side B F is equal to the side B C, because they are two sides of the same square.

3. Wherefore the two sides A B, B F of the triangle A B F are equal to the two sides D B, B C f the triangle D B C.

4. Also the included angles A B F and D B C may be proved equal, thus: the angle F B C is equal to the angle D B A because they are angles of squares, and the angles of squares are right angles (Def. 30), and all right angles are equal (Ax. 11).

And if to each of these equal angles the angle A B C be added, then the whole angle A B F is equal to the whole angle D B C, which are the included angles.

5. Therefore by the fourth proposition the two triangles A B F and D B C are equal in every respect. Wherefore the base A F is equal to the base D C; which was to be proved.

EXERCISE V. (fig. 33).

Describe two circles B D F and C E G from the same centre A, but with different radii. From the common centre of the two circles draw two radii (as in the figure) A C, A E, to the outer circumference, cutting the inner circumference in B and D; join B E, D C:—

It is required to prove that B E is equal to D C; also that the angle A B E is equal to A D C, and the angle A C D to A E B.

Hints.

The two triangles to be compared are A B E and A D C, and the bases of the triangles are B E, D C, these being the straight lines to be proved equal.

Hence it follows that the two sides are D A, A C in the triangle D A C, and B A, A E in the triangle B A E; also that the included angle is the angle at A, which is common to both triangles.

Now, without further hints, try, and for a particular reason that you will know by-and-by, *try hard* to write down the five steps of the demonstration without reading further.

Demonstration.

1. A B is equal to A D, because they are both radii of the same circle B D F.

2. A C is equal to A E, because they are both radii of the same circle C E G.

3. Wherefore the two sides B A, A E of the triangle B A E are equal to the two sides D A, A C of the triangle D A C.

4. And the included angle B A E is equal to the included D A C, both being one and the same angle [or, in other words, the included angle at A is common to both triangles].

5. Therefore by the fourth proposition the two triangles A B E and A D C are equal in every respect; wherefore the base B E is equal to the base D C, and the remaining angles, to which the equal sides are opposite, are equal; viz., the angle A D C to the angle A B E, and the angle A C D to the angle A E B; which was to be proved.

EXERCISE VI. (fig. 34).

The sixth exercise is to be taken in connection with the fifth.

The only additional construction is, join B D; and taking as demonstrated all that was proved in the fifth exercise:—

It is required to prove that the angle C B D is equal to the angle E D B, and that the angle C D B is equal to the angle E B D.

Hints.

The two triangles to be compared in this exercise are, clearly, the triangles C B D and E D B. The next question is, What are the bases of these triangles? They may be discovered by the following consideration (among others):

In Exercise V. it was demonstrated that the angle A C D is equal to the angle A E B, which is the same as saying that the angle B C D is equal to the angle D E B; that is, in the two triangles to be compared, the angle B C D has been demonstrated to be equal to D E B. And since in any two triangles to be compared by the fourth proposition the angles which are given, or are known to be, equal are opposite to the bases, it follows that B D is the base of each of the triangles C B D and E D B. Wherefore the two *sides* of these triangles are B C, C D of the one, and D E, E B of the other. Now, before we can say that by the fourth proposition the triangles C B D and E D B are equal in every respect, we must show that the two sides B C, C D of the one are equal to the D E, E B of the other, each to each. Can we do so?

First. Is B E equal to C D? It is. These lines were proved equal in the fifth exercise; they are the bases of two triangles A D C, A B E, proved equal in every respect by the fourth proposition.

Secondly. Is B C equal to D E? It is; for the whole A C is equal to the whole A E (Def. 15), and A B a part of A C is equal to A D a part of A E (Def. 15); therefore the remainder B C is equal to the remainder D E (Ax. 3).

With the above hints to guide you, try now to write out the five steps of the demonstration before reading further.

Demonstration.

1. The side D C has been demonstrated to be equal to B E (Exercise V.).

2. The side B C is equal to D E, for the whole A C is equal to the whole A E, of which the parts A B and A D are equal; therefore the remainder B C is equal to the remainder D E (Ax. 3).

3. Wherefore the two sides B C, C D of the triangle B C D are equal to the two sides D E, E B of the triangle D E B.

4. And the included angle B C D has been demonstrated to be equal to the included angle D E B (Exercise V.).

5. Therefore by the fourth proposition the two triangles C B D and E D B are equal in every respect; wherefore the remaining angles, to which the equal sides are opposite, are equal; viz., the angle C B D to E D B, and the angle B D C to D B E; which was to be proved.

Deduction from Exercises V. and VI.

This deduction is not an exercise on the fourth proposition. It is introduced here for a special reason, which you will know afterwards.

Enunciation.

It is required to deduce—from the fifth and sixth exercises—that the angle A B D is equal to the angle A D B (fig. 34).

Hints.

The hints for this deduction are given in the form of five suggestive questions. The answers are put in a footnote, and let the learner try to make out the answer himself first, before he looks at the note below.

1. In the fifth exercise, what angle was proved equal to A B E?

2. And in the sixth exercise, what angle was proved equal to D B E?

3. Now, if from the angle A B E the angle D B E be taken, what angle remains?

4. And if from the angle A D C the angle B D C be taken, what angle remains?

5. And if equals be taken from equals, what do we know about the remainders?[1]

Here follows the demonstration in due form. Try to write it out before reading further.

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[1] *Ans.* 1. The angle A D C. *Ans.* 2. The angle B D C. *Ans.* 3. The angle A B D. *Ans.* 4. The angle A D B. *Ans.* 5. That they are equal.

Demonstration.

1. The whole angle A B E has been demonstrated to be equal to A D C (Exercise V.).

2. And the angles D B E and B D C, parts of these have also been demonstrated to be equal (Exercise VI.).

3. Therefore the remaining angle A B D is equal to the remaining angle A D B (Ax. 3). Q.E.D.

LAST EXERCISE (fig. 35).

Before bringing these exercises to a close, one more is introduced to test the learner's knowledge of the fourth proposition, and his power rightly to apply it. The more so as there is a story connected with it.

The exercise is as follows:

Enunciation.

Let B C D be a circle of which A is the centre, and let B C be a straight line not passing through the centre, and let it be granted that A E is drawn perpendicular to B C:—

It is required to prove that B C is bisected in E.

Construction.

Join A B and A C.

Now, this exercise is really the third proposition of the third book of Euclid. And the story connected with it is this, that the question was set in an examination for a very distinguished honour at Cambridge. The candidates were selected men, yet all of them but two attempted to prove it by an incorrect application of the fourth proposition.

The attempted solution of those candidates who did the proposition wrong is here put down; and the learner is strongly advised to try without help to find out their mistake, and why it is a mistake.

The incorrect solution is as follows:

1. The side A B is equal to the side A C because they are radii of the same circle.

2. The side A E is common to the two triangles B A E and A C E.

3. Wherefore the two sides B A, A E of the triangle B A E are equal to the two sides C A, A E of the triangle C A E.

4. And the angle B E A is equal to the angle C E A, because they are right angles (Def. 10).

5. Therefore, by the Fourth Proposition, the two triangles A B E and C A E are equal in every respect. Wherefore B E is equal to E C. Q.E.D.

The learner will do well not to read the following explanation of the mistake till he has first tried to find it out himself.

The error is this, that the angles B E A and C E A are not the angles *contained* by the sides of the triangles which are equal, each to each. The Fourth Proposition proves that two triangles are equal in every respect, if two sides of the one are equal to two sides of the other, each to each, and likewise the *angles contained by those sides are equal.*

Now, the *sides* (see the paragraph marked 3 of the above incorrect solution) are B A, A E of the triangle B A E, and C A, A E of the triangle C A E, and the angles contained by these sides are B A E and C A E respectively; and these angles are *not* given equal (the angles given equal are A E B and A E C), and therefore the triangles cannot be proved equal by the Fourth Proposition. In fact, the equality of B E and C E has to be proved by a subsequent Proposition, namely, the Twenty-sixth.

This concludes the exercises on the Fourth Proposition; and we now pass on to the Fifth, a proposition which bears a name which makes it dreaded by beginners. Those, however, who have come thus far through this treatise need not fear it, for the simple reason that they have already done it. The foregoing Fifth and Sixth Exercises, with the appended deduction from them, is really the Fifth Proposition in a slightly disguised form, as, doubtless, some learners have already surmised.

On the Fifth Proposition.

The enunciation of the Fifth Proposition may be put thus:

It is given that the two sides of a triangle are equal, in other words, that the triangle is isosceles:—

It is required to prove that the angles at the base are equal.

And if the two equal sides of the isosceles triangle be produced:—

It is required to prove that the angles on the other side of the base are equal.

Particular Enunciation (fig. 36).

Let A B C be an isosceles triangle, having the side A B equal to the side A C:—

It is required to prove that the angle A B C is equal to the angle A C B.

Also, if the equal sides A B, A C be produced to any length beyond B and C, say to the points D and E:—

It is required to prove that the angle D B C is equal to the angle E C B.

In order to prove this proposition, Euclid makes a couple of triangles, which he will be able to compare and prove equal in every respect by the Fourth Proposition, in the following manner. In B D he takes any point F, and from A E, the greater, he cuts off, by the Third Proposition, a part A G equal to A F, and he joins F C and B G.

Now, if the learner turns back to Exercise V. (fig. 33), he will recognise in the two triangles A D C and A B E a pair of triangles exactly corresponding to the two triangles A B G and A C F (fig. 36).

And these triangles can be proved equal in every respect, as the two triangles A B E and A D C were proved equal in Exercise V.

This is the first part of Prop. V.

Next, he proves the triangles B C F and C B G equal in every respect by the Fourth Proposition, in the same manner as the triangles D B C and E B C were proved equal in Exercise VI.

This is the second part of Prop. V.

Lastly, he deduces that the angle A B C is equal to the angle A C B, as was done in the deduction appended to the Fifth and Sixth Exercises.

This is the third part of Prop. V.

It will be a great assistance to the learner to notice particularly that the Fifth Proposition falls naturally into three parts, corresponding, respectively, to Exercise V. and VI., and to the deduction appended to those exercises.

And now, with these hints, and references to these foregoing exercises, the learner is recommended boldly to try to write out the Fifth Proposition without professedly learning it. Let him carefully study Exercises V. and VI., and the appended deduction; also the hints given above; and then write out, if he can, the Fifth Proposition as an exercise on the Fourth.

It is almost too much to hope that he will succeed in the endeavour, but it is worth trying, and if he succeeds it will be a great success. However, let him not be discouraged if he fails in his attempt to write out the demonstration without professedly learning the proposition.

If he fails to attain this 'great success,' let him read carefully what follows, viz., the proposition written out in due form. If after doing so he finds that he is able to write out the demonstration, even this will be a very satisfactory amount of success, for many and many a poor fellow has spent weeks, if not months, over this proposition, and at the end of the time his mind has been in a state of utter confusion and bewilderment about it.

Proposition V. Theorem.

General Enunciation.

If a triangle is isosceles:—

It is required to prove that the angles at the base are equal.

And if the equal sides be produced:—

It is required to prove that the angles on the other side of the base are equal.

Particular Enunciation (fig. 36).

Let A B C be an isosceles triangle, having the side A B equal to A C:—

It is required to prove that the angle A B C is equal to the angle A C B.

And if the equal sides A B and A C be produced to D and E:—

It is required to prove that the angle D B C is equal to the angle E C B.

Construction.

The sides A B and A C being produced to D and E, in B D take any point, F, and from A E the greater cut off a part A G equal to A E the less, and join F C, B G.

Demonstration.

PART I.

First, in the two triangles A B G and A C F:

We have:

1. A G equal to A F (construction).
2. A B equal to A C (hypothesis).
3. Wherefore the two sides F A, A C of the

triangle F A C are equal to the two sides G A, A B of the triangle G A B, each to each.

4. And the included angle at A is common to the two triangles F A C and B A G.

5. Therefore, by Prop. IV., the two triangles F A C and B A G are equal in every respect. Wherefore the base B G is equal to the base F C, and the remaining angles, to which the equal sides are opposite, are equal, viz. the angle A B G to the angle A C F, and the angle A F C to the angle A G B.

PART II.

Secondly, in the two triangles F B C and G C B:

We have:

1. The side F C equal to the side B G. It has been proved so above.

2. The side F B is equal to the side G C, as may thus be shown:

The whole A F is equal to the whole A G (cons.), parts of which, namely, A B and A C, are also equal (hyp.).

Therefore the remainders F B and G C are equal (Ax. 3).

3. Wherefore the two sides C F, F B of the triangle F C B are equal to the two sides B G, G C of the triangle G B C, each to each.

4. And the included angle B F C has been proved to be equal to the included angle C G B.

5. Therefore, by Prop. IV., the two triangles F C B and G B C are equal in every respect. Wherefore the remaining angles, to which the equal sides are opposite, are equal, viz. the angle F B C to the angle G C B, and the angle B C F to the angle C B G.

PART III.

1. Now the whole angle A B G was proved, in Part I., to be equal to the whole angle A C F.

2. Parts of which, viz. the angle C B G, and the angle B C F, were proved equal, in Part II.

3. Therefore the remaining angle A B C is equal to the remaining angle A C B (Ax. 3). And these are the angles at the base of the isosceles triangle.

4. Also the angle F B C has been shown (Part II.) to be equal to the angle G C B. And these are the angles on the other side of the base.

Wherefore, if a triangle is isosceles, the angles at the base are equal; and if the equal sides be produced, the angles on the other side of the base are also equal. Q.E.D.

From the above proposition may easily be deduced the corollary[1] that every equilateral triangle is also equiangular.

Let A B C (fig. 37) be an equilateral triangle:—

It is required to prove that it is equiangular.

Demonstration.

1. Any two of its sides (as A B and A C) are equal; therefore, by Prop. V. the angle A B C is equal to the angle A C B.

2. Also, any other two of its sides (as A B and B C) are equal; therefore, by Prop. V., the angle B A C is equal to the angle B C A.

3. Also the two other sides, A C, C B, are equal; there-

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[1] Whatever may be obviously gathered or deduced from a proposition is called a '*corollary*' to it. *Corollarium* meant originally a garland or wreath of thin metal given as a reward. Perhaps the idea of a wreath or garland, hanging from its support, caused the word to be used later, by philosophical writers to express a deduction, an inference.

fore, again by Prop. V., the angles C A B and C B A are equal.

Therefore all the angles are equal. Q.E.D.

[Of course, having proved that any two angles of the equilateral triangle are, each of them, equal to the third angle, as done in 1 and 2, it might be inferred by Axiom 1 that they are equal to one another.]

On the Sixth Proposition.

The general enunciation of the Sixth Proposition is as follows:

Given that two angles of a triangle are equal to each other:—

It is required to prove that the sides also which subtend or are opposite to the equal angles, shall be equal to one another.

Particular Enunciation.

Let A B C (fig. 38) be a triangle having the angle A B C equal to the angle A C B:

It is required to prove that the side A B (opposite to the angle A C B) is equal to the side A C (opposite the angle A B C).

This proposition is what is called a *converse* proposition. It is the converse of the Fifth. By writing out the enunciations of the two propositions, the Fifth and Sixth, in juxtaposition, you will see the meaning of the word 'converse.'

In the Fifth Proposition it is said:

If the side A B is equal to the side A C: The angle A B C is equal to the angle A C B.

In the Sixth Proposition it is said:

If the angle A B C is equal to the angle A C B: The side A B is equal to the side A C.

What is given in the Fifth has to be proved in the Sixth, and what is given in the Sixth has to be proved in the Fifth —in other words, the premises (what is given) and the conclusion (what has to be proved) change places.

Now very few converse propositions admit of a direct proof. Euclid cannot prove directly that, if the angle A B C is equal to the angle A C B, the side A C is equal to the side A B. But he *can* and *does* prove that it is false to say that A C and A B are unequal; and hence he infers that they are equal.

To appreciate this method of proof, let us consider Euclid as forcing conviction on an antagonist who denies what Euclid contends is true.

Imagine this antagonist to say: 'I deny that A B is equal to A C.'

Euclid replies: 'If it is *as you say*—i.e., if AB is not equal to A C—one of them must be greater than the other.'

The opponent is obliged to admit this.

'Well, then,' says Euclid, 'suppose it is A B that is the greater, then we can, by Prop. III., cut off from A B the greater a part equal to A C the less.'

The opponent must admit this also.

'Suppose it done,' says Euclid, 'that is, suppose B D to be cut off from B A,—which *you say* is the greater,—equal to A C the less.'

'Granted,' says the opponent.

'Then,' adds Euclid, 'we will join C D.'

The opponent must allow him to do so: it is the first postulate.

'Now then,' says Euclid, 'I have before me two triangles A C B and D B C, the bases of which are A B and D C respectively; and these triangles I can prove equal in every respect by the Fourth Proposition, as follows:

1. The side D B of the triangle D B C has been cut off from A B [which the antagonist said was the greater] so as to be equal to the side A C of the triangle A C B.

2. The side B C is common to both triangles.

3. Wherefore the two sides D B, B C of the triangle D B C are equal to the two sides A C, C B, of the triangle A C B.

4. And the angle D B C, contained by the two sides D B,

B C, is *by the hypothesis*, equal to the angle A C B contained by the two sides A C, C B.

5. Therefore, by Prop. IV., the two triangles A B C and D B C are equal in every respect, and therefore in respect of area. That is, the space enclosed by the three straight lines D B, B C, C D, is equal to the space enclosed by the three straight lines A B, B C, C A. But this first space is only a part of the second, and a whole is greater than a part (Axiom 9); therefore the conclusion that they are equal is absurd.'

[It is to be hoped that no one who has followed Euclid's reasoning so far will here be disposed to make the common, but weak observation, What is the use of proving what is absurd? but will rather see that Euclid has now completely 'shut up' his opponent. The opponent had said A B and A C are unequal. Euclid answers, For the sake of argument, suppose it to be as you say, and see what comes of it. And what *does* come of it? Why this, that if what the opponent asserts is true, a part can be proved equal to a whole. But a part cannot equal a whole, therefore what the opponent asserts, viz., that A B, A C are unequal, is false,[1] and all that Euclid has to add, is:]

'Therefore A B is not unequal to A C, that is, it is equal to it. Which was to be proved.'

Perhaps some learners will be able, with the above explanation, to write out the proposition, as is done below, without reading further for the present.

PROPOSITION VI. THEOREM.

General Enunciation.

It is given that two angles of a triangle are equal to each other:—

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[1] An absurd conclusion must result, either from the hypothesis (the foundation of the reasoning) being false, or from a flaw in the reasoning itself. It is hoped that the pupil's knowledge of the Fourth Proposition, and his power of rightly applying it, is now so confirmed that he will without hesitation decide:—'It is the hypothesis that is false.

It is required to prove that the sides also which subtend, or are opposite to, the equal angles are also equal.

Particular Enunciation.

Let A B C (fig. 38) be a triangle having the angle A B C equal to the angle A C B:—

It is required to prove that the side A C shall be equal to the side A B.

Construction.

If A B be not equal to A C,

One of them must be greater than the other.

Suppose A B to be the greater.

Then from it cut off a part B D equal to A C the less (Prop. 3).

And join D C.

Demonstration.

Then in the triangles D B C and A C B we have:

1. D B equal to A C (they were made so on the assumption that A B and A C were unequal).
2. B C is common to both triangles.
3. Wherefore the two sides A C, C B of the triangle A C B, are equal to the two sides D B, B C of the triangle D B C.
4. And the included angle D B C is by the hypothesis equal to the included angle A C B.
5. Therefore, by Prop. IV., the two triangles D B C and A C B are equal in every respect, and therefore in respect of area; that is, the enclosed space D B C is equal to the enclosed space A C B. But D B C is only a part of A B C, and therefore cannot be equal to it (Ax. 9).

Therefore A C is not unequal to A B, that is, it is equal to it.

Wherefore, if two angles of a triangle be equal to one another, the sides also which are opposite to the equal angles are equal. Q.E.D.

Corollary. Hence it may be deduced that every equiangular triangle is also equilateral.

The truth of this corollary is deduced from the Sixth Proposition precisely as the former corollary (the converse of this) was deduced from the Fifth Proposition.

On the Seventh and Eighth Propositions.

These two propositions must be taken together. The eighth is as follows: If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal; then also must the angle contained by the two sides of the one be equal to the angle contained by the two sides equal to them of the other. It will be seen by this enunciation that the Eighth Proposition is the converse of the Fourth. And, like the Fourth, it is a very important proposition. The Seventh is only a subsidiary proposition, introduced in order to prove the Eighth.

The proof which Euclid gives of the Eighth is akin to that which he gave of the Fourth.

He proved in the Fourth that the bases were equal by showing that, if one triangle were applied to the other, as directed, the bases would fill the same space; so, in the Eighth, his object is to show that if one triangle be applied to the other, as he directs, that the included angles will fill the same space.

But this he cannot do without the help of the Seventh Proposition.

Now this proposition, it must be confessed, is one which commonly gives very much trouble to learners, so much so that it is often given up, and another proof of the Eighth Proposition is substituted instead of Euclid's.

But as the main difficulty of the Seventh Proposition is a piece of geometrical reasoning which occurs in many propositions, and must be got over sooner or later, it may be as well to do so at once.

The following very homely illustration may serve to explain and simplify the argument, which in its purely geometrical form often perplexes.

There is a family of four brothers (see fig. 39); the two middle ones are twins, the counterparts of each other, both being of exactly equal height. We will call the twins No. One and No. Two. There is an elder brother who is known to be taller than twin No. One, and there is a younger brother who is known to be shorter than twin No. Two. Euclid wants to prove that the elder brother is taller than the younger brother. He does it in this way:

We know, he says, that twin No. Two is taller than the younger brother.

But twin No. One is equal to twin No. Two.

Therefore twin No. One is taller than the younger brother.

But the elder brother is taller than twin No. One.

Much more, he concludes, the elder brother is taller than the younger.

Now this, doubtless, seems very simple. The learner is recommended to fix this illustration in his mind, and, when he comes to the Seventh Proposition, to note which angles represent the twins, and which the elder, and which the younger brother.

The enunciation of the Seventh Proposition is as follows:

Upon the same base and on the same side of it there cannot be two triangles which have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity. In other words: If upon the same base and on the same side of it there be two triangles, and it be given that the two sides terminated in one extremity of the base are equal, it is required to prove that the sides terminated in the other extremity cannot be equal.

Particular Enunciation.

Let A B C and A B D be two triangles (fig. 40), standing on the same base A B, and on the same side of it, and let it be given that A C is equal to A D :—

It is required to prove that the side B C cannot be equal to the side B D.

It is clear that the triangles in question must assume one of three different positions.

Either, as in fig. ($40_{(1)}$), the vertex D of one of the triangles may be outside the other triangle.

Or, as in fig. ($40_{(2)}$), the vertex D may be within the other triangle.

Or, as in fig. ($40_{(3)}$), the vertex D may be on the side of the other triangle.

We will take each case separately.

CASE 1.

In which the vertex D of the triangle A B D falls outside of the triangle A C B.

Construction.

Join C D.

Demonstration.

1. Then, because A C is equal to A D (hyp.), we know by the Fifth Proposition, that the angle A C D is equal to the angle A D C. From hence Euclid will be able to prove that the angle B D C is greater than the angle B C D.

Referring to the illustration above given, the angles A D C and A C D represent, respectively, the two twins; A D C being twin No. One, and A C D twin No. Two. The younger brother is B C D, for A C D (twin No. Two) is known by Ax. 9 to be greater than B C D (the younger brother); also B D C is the elder brother, for B D C is known by the same axiom to be greater than A D C (twin No. One).

Now, the reasoning given above in reference to the brothers may thus be written out in reference to the angles which stand for them.

1. The angle A C D [twin No. Two] is equal to the angle A D C [twin No. One].
2. But the angle A C D [twin No. Two] is greater than the angle B C D [the younger brother].
3. Therefore also the angle A D C [twin No. One] is greater than the angle B C D [the younger brother].
4. But the angle B D C [the elder brother] is greater than the angle A D C [twin No. One].
5. Much more then the angle B D C [the elder brother] is greater than the angle B C D [the younger brother].

Euclid thus proves that if A C is equal to A D, the angle B D C must be greater than the angle B C D. He then carries on the reasoning thus:

If B D were equal to B C, it would follow from Prop. V. that the angle B D C must be equal to B C D.

Now angles cannot be unequal and equal at the same time.

Therefore, if A C be equal to A D, B C cannot be equal to B D.

That is, we have proved in Case 1 that on the same base A B, and on the same side of it, there cannot be two triangles which have the two sides A C and A D ending in the extremity A equal to each other, and *at the same time* the sides B C and B D, ending in the other extremity, equal to each other.

CASE 2.

We have now to consider Case 2, in which the vertex D of the triangle A D B falls within the triangle A C B (fig. $40_{(2)}$).

Construction.

In this case, besides joining the points C and D, produce the sides A C and A D to E and F.

Demonstration.

Here also the angle E C D is equal to the angle F D C by the Fifth Proposition, for they are the angles on the other side of the base of the isosceles triangle A C D.

These equal angles then are the twins: F D C is twin No. One, and E C D twin No. Two. Also B D C is the elder brother, for B D C is known (Ax. 9) to be greater than F D C (twin No. One). And B C D is the younger brother, for, by the same axiom, B C D is known to be less than E C D (twin No. Two).

The learner is earnestly advised here to close the book, and, with the fig. $40_{(2)}$, before him (keeping in mind, if necessary, the illustration of the four brothers), to write out the five steps of the reasoning by which Euclid proves that the angle B D C is greater than the angle B C D.

As an encouragement, he may be told that many beginners, after spending days and even weeks over this proposition, have entreated that they might be excused learning the second part, and have begged that it 'might be the first case that should be set in the examination.'

Here follow the five steps, above alluded to, of Euclid's reasoning, to prove, in the second case, that the angle B D C is greater than the angle B C D.

1. The angle E C D is equal to the angle F D C (Prop. 5).
2. But the angle E C D is greater than the angle B C D (Ax. 9).
3. Therefore also the angle F D C is greater than the angle B C D.
4. But the angle B D C is greater than the angle F D C (Ax. 9).
5. Much more, then, is the angle B D C greater than the angle B C D.

From this Euclid proceeds to reason exactly as in Case 1. If B D were equal to B C, the angle B D C would *be equal* to the angle B C D.

But, on the supposition that A C is equal to A D, it has been proved that the angle B D C is *greater* than the angle B C D.

And angles cannot be equal and unequal at the same time.

Therefore, if A C is equal to A D, B D cannot be equal to B C.

That is, we have proved in Case 2, as we did in Case 1, that on the same base, &c.

CASE 3.

The third case, where the vertex D falls *on* the side of the triangle A C B, Euclid says needs no demonstration. It may be easily seen (fig. 40(3)) that B C cannot be equal to B D, because a whole is greater than a part.

So that it it evident in Case 3, also, as well as in Cases 1 and 2, that on the same base, &c.

We have thus proved that, *in no case*, can two distinct, separate triangles stand on the same base, and on the same side of it, which have the two sides ending in one extremity of the base equal to each other, and likewise the two sides ending in the other extremity of the base. Q.E.D.

The use which Euclid makes of this proposition in order to prove the Eighth, will be shown by and by. In the meantime it will be well here to write out the Seventh Proposition in its purely geometrical form, without any reference to the four brothers.

PROPOSITION VII. THEOREM.

General Enunciation.

On the same base, and on the same side of it, there cannot be two triangles which have the two sides terminated in one extremity of the base equal to each other, and likewise the two sides terminated in the other extremity.

Particular Enunciation.

On the same base A B (see fig. 40), and on the same side of it, there cannot be two triangles which have the two sides A C and A D, terminated in the extremity A, equal to each other, and *likewise* the two sides B C and B D terminated in the extremity B.

First, taking the case in which the vertex of each of the triangles A C B, A D B is without the other triangle (fig. 41(1)).

Construction.

Join the two vertices C and D.

Demonstration.

1. On the supposition that A C is equal to A D, the angle ACD is equal to the angle ADC (Prop. 5).
2. But the angle A C D is greater than the angle BCD (Ax. 9).
3. Therefore also the angle A D C is greater than the angle B C D.
4. Much more is the angle B D C greater than the angle B C D.
5. On the other hand, on the supposition that B C is equal to B D, the angle B D C would be equal to the angle B C D (Prop. 5).

While on the supposition that A C is equal to A D, these angles are shown above to be unequal.

Therefore A C cannot be equal to A D, and *likewise* B C equal to B D, if the vertex of one of the triangles fall *without* the other.

Secondly, taking the case in which the vertex D, of one of the triangles A D B, is within the other triangle A C B (fig. 41(2))

Construction.

Besides joining the vertices C, D (fig. $40_{(2)}$), produce the two sides A C and A D to E and F.

Demonstration.

On the supposition that A C is equal to A D,

The angle E C D is equal to the angle F D C (Prop. 5).

But the angle E C D is greater than the angle B C D (Ax. 9).

Therefore also the angle F D C is greater than the angle B C D.

Much more is the angle B D C greater than the angle B C D.

On the other hand, on the supposition that B C is equal to B D, the angle B D C would be equal to the angle B C D (Prop. 5).

While on the supposition that A C is equal to A D, these angles are shown above to be unequal.

Therefore A C cannot be equal to A D, and *likewise* B C equal to B D, in the case in which the vertex of one of the triangles is *within* the other.

Thirdly, taking the case in which the vertex D of one of the triangles A D B is on the side of the other triangle A C B (fig. $40_{(3)}$).

This case needs no demonstration, for it is evident from Ax. 9 that B C cannot be equal to B D.

Therefore in no case on the same base and on the same side of it can there be two triangles which have the two sides terminated in one extremity of the base equal to each other, and likewise the two sides terminated in the other extremity. Q.E.D.

We now proceed to apply the Seventh Proposition to prove the Eighth, the enunciation of which, though already given, may be here repeated.

It being given that two triangles have the two sides of the one equal to the two sides of the other, each to each, and have likewise their bases equal:—It is required to prove that the angle which is contained by the two sides of one of the triangles is equal to the angle which is contained by the two sides equal to them of the other triangle.

Let A B C and D E F (fig. 41$_{(1)}$) be two triangles which have the two sides B A, A C of the one equal to the two sides E D, D F of the other, each to each, viz., B A to E D, and A C to D F, and let them likewise have the base B C equal to the base E F:—

It is required to prove that the angle B A C is equal to the angle E D F.

This is done as follows:

Conceive, Euclid says, the triangle A B C to be placed on the triangle D E F [not here, as in the Fourth, so that A rests on D, but] so that B rests on E, and the base B C lies along E F. Then the point C shall come on the point F, because B C is equal to E F.

Thus the two triangles A B C and D E F are now standing on one and the same base, which may be called either B C or E F, as these two straight lines now form one and the same straight line. Observe in the figure the first triangle A B C has a few strokes drawn across it, and the double letters B, E and C, F are placed at the extremities of the base; this is done to remind the learner that the triangle A B C is no longer to be considered as lying in its original position. Its base B C is now to be conceived as applied to and lying along the base E F, so that the two bases form the common base of both the triangles.

Also the learner is to keep in mind that the two sides B A and E D, which both terminate in one extremity of the base, are equal to each other, and that the sides C A and

F D, which both terminate in the other extremity of the base, are also equal to each other.

And this being so, the question is, where will the two sides B A, A C fall?—will they fall exactly on E D, D F, or will they not? The first impression of every one would, doubtless, be that B A, A C *would* fall on E D, D F, and Euclid *proves* by the Seventh Proposition that such an impression would be correct.

He says: If B A, A C do *not* fall on E D, D F, they must lie in some other direction, as, for instance, in the direction B A, A C, indicated in the figure 41(1). And *if* they do so, then we *shall* have upon the same base, and on the same side of it, two separate triangles having the two sides B A and E D terminating in one extremity of the base equal to each other, and *likewise* the two sides C A and F D terminating in the other extremity of the base equal to each other. But this was proved to be impossible in the Seventh Proposition.

So that B A, A D cannot lie in the direction indicated in the figure. That is, they can lie nowhere but on E D, D F. And this being so, the space filled by the angle E D F, and that filled by B A C, is the same space; and as the two angles fill the same space, they are equal (Ax. 8).

[In fact, B A, A C, lying on E D, D F, if the little curved rim spoken of (p. 6) to indicate the angle B A C, and E D F, were drawn, it would have to be drawn through exactly the same space.]

This, then, is the nature of Euclid's proof, and the way in which he uses the Seventh Proposition to prove the Eighth. It only remains now to write out the Eighth Proposition formally, without explanatory remarks.

PROPOSITION VIII. THEOREM.

General Enunciation.

It being given that two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal:—

It is required to prove that the angle contained by the two sides of the one is equal to the angle contained by the two sides equal to them of the other.

Particular Enunciation.

Let A B C and D E F be two triangles (fig. 41(2)), which have the two sides B A, A C of the one equal to the two sides E D, D F of the other, each to each, viz. B A to E D, and A C to D F; and have likewise the base B C equal to the base E F:—

It is required to prove that the included angle B A C is equal to the included angle E D F.

Demonstration.

Let the triangle A B C be applied to the triangle D E F, so that the point B may be on E, and the base B C on E F.

Then shall the point C come on the point F, because B C is equal to E F.

And the two triangles A B C and D E F will stand upon the same base E F, and on the same side of it.

This being the case, the two sides B A, A C must lie along the two sides E D, D F, for if not they will have a different situation as E G, G F.

And then, on the same base and on the same side of it, there *will* be two triangles which have the two sides terminated in one extremity of the base equal to each other, and likewise the two sides terminated in the other extremity, which is impossible (Prop. 7).

Therefore the two sides B A, A C cannot but lie on the two sides E D, D F, and therefore the angles B A C and E D F fill the same space.

And therefore they are equal (Ax. 8).
Wherefore, if two triangles, &c. Q.E.D.

Euclid might have proceeded to show that the two remaining angles are also equal, as well as the included areas, for he has here shown, as completely as he did in the Fourth Proposition, that the two remaining angles and the included areas *fill the same space.* He does not, however, proceed any further, nor is it necessary, for when he has proved that the included angles are equal, he has before him two triangles which have two sides of the one equal to two sides of the other, and likewise the included angles equal: therefore, by the Fourth Proposition, the two triangles are equal in every respect.

Allusion was made (page 74) to a proof of the Eighth Proposition, independent of the Seventh, which is given in many schools instead of Euclid's proof. This proof is as follows. It is an easy deduction from the Fifth Proposition.

As in Euclid's proof, the triangle A B C is applied to the triangle D E F, so that the point B lies on the point E, and the base B C on the base E F, and, it is said, the point C will come on the point F, because the bases are equal. So far as in Euclid. But then, instead of the two triangles lying on the same side of the common base, the triangle A B C is supposed to lie on the opposite side of the base, in the position E G F (fig. 42).

Construction.

Join D G.

And first, let the line D G intersect the base E F (fig. 42(1)).

Demonstration.

We have E G equal to E D (hyp.).

And therefore the angle E G D equal to the angle E D G (Prop. 5).

Also F D is equal to F G (hyp.).

And therefore the angle F D G is equal to the angle F G D (Prop. 5).

Therefore the whole angle E D F is equal to the whole angle E G F (Ax. 2), that is, to the angle B A C. Q.E.D.

In the case in which the line D G joining the vertices of the triangles intersects the base produced (as in fig. 42(2).)

It may be proved, as in Case 1, that the angle E D G is equal to the angle E G D, and that the angle F D G is equal to the angle F G D.

And therefore the remaining angle E D F is equal to the remaining angle E G F (Ax. 3), that is, to the angle B A C. Q.E.D.

If the straight line D G joining the vertices of the triangle passes through the point F (as in fig. 42(3)), the demonstration is simply as follows:

Then is the angle E D G equal to the angle E G D Prop. 5); in other words, the angle E D F is equal to the angle E G F, that is, to the angle B A C. Q.E.D.

On the Ninth Proposition.

The learner having proved the Eighth Proposition as well as the Fourth, can now go forward with these two propositions, one in either hand, engines, as it were, of use and power in the solution of other problems and theorems. The next four propositions are problems, to be solved by one or other of these two propositions. But the learner must be careful to apply them rightly. If it is the included angles which have to be *proved* equal, he must apply the Eighth Proposition. If it is the bases which have to be *proved* equal, he must apply the Fourth Proposition.

In both cases the first three of the five steps of reasoning, often repeated in the exercises on the Fourth Proposition, are the same, whether it is the Fourth or the Eighth Proposition which is applied. The last two steps change places.

In applying the Fourth Proposition, the fourth and the fifth steps of the reasoning are,

(4.) And the included angles [naming them] are equal [the reason why these angles are known to be equal being added]. And the conclusion is:

(5.) Therefore by the Fourth Proposition the bases are equal. Q.E.D.

In applying the Eighth Proposition, the last two steps of the reasoning are as follows:

(4.) And the bases are equal [the reason why they are known to be equal being added].

And the conclusion is:

(5.) Therefore, by the Eighth Proposition, the included angles are equal. Q.E.D.

With these preliminary remarks we turn to the Ninth Proposition, which is as follows:

Given a rectilineal angle:—

It is required to bisect it.

Euclid draws an angle (see fig. 43(1)), and calls it the angle B A C, and adds, It is required to bisect B A C.

His construction is as follows:

In A B he takes any point D.

And by the Third Proposition he cuts off from A C a part A E equal to A D.

He joins D E.

And on the side of D E, remote from A, he describes an equilateral triangle D F E by Prop. I.

[You will be told by and by why he says on the side of D E *remote* from A].

He adds, Join A F.

And he concludes thus:

Then will the angle B A C be bisected by the straight line A F.

Before writing out the demonstration of this proposition in a formal manner, the following suggestions are given, to encourage the learner to make out the demonstration for himself.

1. We have to prove that the angle B A C is bisected. That is, we have to prove that the angle B A F is equal to the angle C A F.

2. As it is 'angles' and *not* 'bases' that have to be proved equal, it is the Eighth Proposition that must be used.

3. *Probably* the two triangles to be compared by the Eighth Proposition are A D F and A E F, and we shall know that they *certainly* are, if we can show that these triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the bases equal.

4. Now which are the bases? Clearly D F and E F, for these are the sides of the triangles A D F and A C F, which are opposite to the angles D A F and E A F which have to be proved equal.

5. And D F, E F being the bases, the two sides of the triangle D A F will be D A and A F, and the two sides of the triangle E A F will be E A and A F. And the angles contained by these sides (or the included angles) will be D A F and E A F.

6. Can we show that the two sides here named of the triangle D A F are equal to the two sides named of the triangle E A F, and that the bases are equal? For an answer to this question we must look back to the construction. The learner, calling to mind the observation made page 29, will remember that since in the construction he was told to make A E equal to A D, he is sure to meet in the demonstration 'A E is equal to A D because they were made so.' Also since in the construction he was told, on D E, to describe an equilateral triangle, he is sure to meet in the demonstration the sentence—that certain straight lines are equal because they are the sides of an equilateral triangle. With these suggestions the learner is advised, without reading further as yet, to write out the five steps of the demonstration by which it is proved that the included angles D A F and E A F are equal, in other words, that the angle B A C is bisected by the straight line A F.

It may appear to some learners that this is a very tedious way of getting up the proposition. They may think that it would be much shorter just to read over the demonstration formally written out. It may be so with some minds, but not with all. These suggestions, which appear to the learner long when written out formally, will, after a little practice, pass through his mind instantaneously, one may say, when he has before him two triangles to be compared; and certainly a little self-reliance and independent thought are of the highest value in mastering the propositions of Euclid. The learner who follows the writer's advice may compare his demonstration with that which is given below.

PROPOSITION IX. PROBLEM.

General Enunciation.

A rectilineal angle being given:—
It is required to bisect it.

Particular Enunciation.

Let B A C (fig. 43(2)) be a given rectilineal angle:—
It is required to bisect it.

Construction.

1. In A B take any point D.

2. From A C the greater cut off A E equal to A D the less (Prop. 3).

3. Join D E.

4. On the side of D E remote from A describe an equilateral triangle D F E.

5. Join A F.

Then will the angle B A C be bisected by the straight line A F.

Demonstration.

1. The side A E is equal to the side A D, because they were made so.

2. The side A F is common to both triangles.

3. Wherefore the two sides D A, A F of the triangle D F A are equal to the two sides E A, A F of the triangle E F A, each to each.

4. And the base D F is equal to the base E F, because they are sides of an equilateral triangle.

5. Therefore by the Eighth Proposition the included angle D A F is equal to the included angle E A F, that is, the angle B A C has been divided into two equal angles D A F and E A F: which was to be done.

The reason why Euclid says, 'On the side of D E *remote* from A, describe the equilateral triangle D F A,' is as follows:

The three straight lines D A, A E, E D (see fig. 43), might *possibly* be all equal, that is, D A E *might* be an equilateral triangle. In this case if the equilateral triangle D F E were described on the same side of D E on which the point A is, the vertex F of this triangle would fall exactly on A. In this case there would be no straight line A F.

Also if the angle B A C (fig. 44(1)) were such that it would fall within the equilateral triangle D E F, when described on the same side of E D as the point A, the line A F would not, in this case, bisect the angle B A C. It would lie outside the angle to be bisected, and in order to bisect the angle B A C, F A must be produced to some point G; and we cannot *at present* prove that D A G is equal to E A G: this is proved by the Thirteenth Proposition.

In all other cases, except these, the equilateral triangle D E F might be described on the same side of D E as the point A (see fig. 44(2)). It may be a useful exercise with the figure 44(2), to prove, as in Prop. IX., that D A F is equal to E A F.

On the Tenth Proposition.

The Tenth Proposition is as follows:

Given a straight line:—

It is required to bisect it.

In the Ninth Proposition Euclid taught us how to bisect *an angle.* He now in the Tenth Proposition shows that, being able to bisect an angle, we can, by doing so, bisect a straight line.

He says, Let A B (fig. 45(1)) be a straight line:—

It is required to bisect it.

He begins the construction by describing an equilateral triangle upon A B.

Then, referring us to the Ninth Proposition, he says, Bisect the angle A C B. [Do not say triangle.]

In order to do this, we should have in C A to take any point E, and from C B to cut off C F equal to C E. Then we should have to join E F, and on the side of E F remote [1] from C we should have to describe an equilateral triangle E F G; we should then, by joining C G, bisect the angle A C B.

The learner is advised to do all the construction here described, in pencil [the pencil lines are represented in the figure by dotted lines], then let him 'ink in' the line C G, and, when the ink is dry, let the pencil marks be effaced, and there will remain only the straight line C G bisecting the angle A C B. Then let him carefully produce the bisecting line C G, if necessary, till it intersects the given straight line in D.

Euclid then says, The straight line A B is bisected in D.

Suggestive Hints for the Demonstration.[2]

These will be very similar to those given in the Ninth Proposition.

In order to prove that the given straight line A B is bisected in D, we must show that A D is equal to D B; and as it is *bases*, not angles, which are to be proved equal, it is the Fourth Proposition that has to be used; and most probably the two triangles to be here compared by the Fourth Proposition are the triangles A C D and B C D.

---

[1] This is a case Euclid provides for by saying, in the Ninth Proposition, 'On the side of D E remote from A;' for if the figure is carefully drawn you will find (what you will be able to prove hereafter) that A B C being an equilateral triangle, C E F is also equilateral, so that if the triangle E G F were described on the same side of E F as C, the point G would come exactly on C.

[2] If any learner feels that he does not need these hints, he can pass them over, and without their help write out the five steps of the demonstration. He can then compare his demonstration with that given below, where the Tenth Proposition is fully written out.

If so, what are their bases? Clearly A D and D B, these being the straight lines which have to be proved equal; and knowing the bases, we know the two sides, and the angles contained by those sides. All then that is required for the demonstration is to show that the two sides A C and C D of the triangle A C D are equal to the two sides B C, C D of the triangle B C D, each to each, and that the angles contained by these sides are equal.

To find if they are so, the learner will have to look back to the construction. He will find that he is there told on A B to describe an equilateral triangle, therefore he may be sure that in the demonstration he will have to say, of certain straight lines, 'they are equal because they are sides of an equilateral triangle.' Also he was told, by the Ninth Proposition, to bisect the angle A C B, therefore he may be sure that he will have to set down in the demonstration, 'A C D is equal to B C D because the angle A C B was bisected by C D.'

With these hints, or without them, the learner is advised here, without reading further at present, to write down the five steps of the demonstration by which it is proved that A B is bisected in D.

The Tenth Proposition fully written out is as follows:

PROPOSITION X. PROBLEM.

General Enunciation.

A straight line being given:—
It is required to bisect it.

Particular Enunciation.

Let A B be a straight line (fig. 45(a)):—
It is required to bisect it.

Construction.

On A B describe the equilateral triangle A B C.

Bisect the angle A C B (Prop. IX.) by the straight line C D intersecting the straight line A B in the point D.

Then will A B be bisected in the point D.

Demonstration.

1. The side A C is equal to the side B C because they are both sides of an equilateral triangle.

2. The side C D is common to both triangles.

3. Wherefore the two sides A C, C D of the triangle A C D are equal to the two sides B C, C D of the triangle B C D, each to each.

4. And the included angle A C D is equal to the included angle B C D, because the angle A C B is bisected by the straight line C D.

5. Therefore by the Fourth Proposition the base A D is equal to the base D B, that is, the straight line A B is bisected in the point D. Q.E.F.

On the Eleventh Proposition.

The Eleventh Proposition is as follows:

Given a straight line and a point in it:—

It is required from the given point to draw a straight line at right angles to the given straight line.

Let A B (fig. 46(a)) be the given straight line, and C the given point in it:—

It is required from the point C to draw a straight line at right angles to A B.

Euclid will want, in order that he may be able to solve this problem, two points in the straight line A B, equally distant from C. Therefore, he says, In A C take any point D, and from C B cut off (by Prop. III.), C E equal to C D.

Then, on the part of the given straight line intercepted between D and E, he describes an equilateral triangle D F E.

And upon joining the vertex F with the given point C, he says C F is at right angles at A B.

It will be necessary here to call to mind the tenth definition, in which it is said, When one straight line falling on another straight line makes the adjacent angles equal, each of them is a right angle.

In order then to prove, in conformity with this definition, that C F is at right angles to A B, all that is required is to prove that the angles F C D and F C E are equal; for if they are equal, each of them is a right angle.

With this to guide him, let any learner, who is able, now write out the five steps of the demonstration by which it is proved that the angle F C D is equal to the angle F C E.

For the help of those who cannot, the hints given in the Ninth Proposition are here repeated. They are here given in the form of suggestive questions, the correct answers to which are given in a foot-note, which it is hoped the learner will not consult till he has written out his own answers to the questions.

1. What have you here to prove equal, 'bases' or 'included angles'?

2. Then what Proposition has to be used. The Fourth or the Eighth?

3. What are the two triangles to be compared?

4. What are their bases?

5. And what therefore are the two sides, and what is the included angle of each of the triangles which have to be proved equal?

6. In the construction you were told to make C E equal to C D: what sentence then are you sure of having in the demonstration?

7. Also you were told on D E to describe an equilateral triangle: what sentence on this account are you certain to have in the demonstration? [1]

---

[1] *Ans.* 1. Included angles. *Ans.* 2. The eighth. *Ans.* 3. C D F, and C E F. *Ans.* 4. D F and E F, these straight lines being opposite to the

With these hints (if he has not done it without them) the learner is advised to write out the five steps of the demonstration, by which it is proved that the angle F C D is equal to the angle F C E.

The Eleventh Proposition fully written out is as follows:

PROPOSITION XI. PROBLEM.

General Enunciation.

A straight line being given, and a point in it:—

It is required from the given point to draw a straight line at right angles to the given straight line.

Particular Enunciation.

Let A B be a given straight line, and C a given point in it (fig. 46(2)) :—

It is required from C to draw a straight line at right angles to A B.

Construction.

In A C take any point D, and from C B (by Prop. III.) cut off C E equal to C D.

On D E describe an equilateral triangle.

Join F C.

Then shall F C be at right angles to A B.

Demonstration.

1. The side C E is equal to the side C D because it has been made so.
2. The side C F is common to both triangles.

angles to be proved equal. *Ans.* 5. D C, C F are the two sides of the triangle D C F, and E C, C F are the two sides of the triangle E C F; the included angles are D C F and E C F. *Ans.* 6. We shall have the sentence 'C E is equal to C D, because it was made so.' *Ans.* 7. We shall have the sentence 'Two straight lines are equal (most probably F D and F E), because they are the sides of an equilateral triangle.'

3. Wherefore the two sides D C, C F of the triangle D C F are equal to the two sides E C, C F of the triangle E C F, each to each.

4. And the base D F is equal to the base E F, because they are the sides of an equilateral triangle.

5. Therefore by the Eighth Proposition the included angle D C F is equal to the included angle E C F.

And these are adjacent angles.

6. But when a straight line standing on another makes the adjacent angles equal, each of them is a right angle (Def. 10).

7. Therefore each of the angles D C F and E C F is a right angle, that is, F C is at right angles to A B.

And it is drawn from the point C. Q.E.F.

On the Twelfth Proposition.

The enunciation of the Twelfth Proposition is as follows:

Given a straight line of unlimited length, also a point without it:—

It is required from the given point to draw a straight line perpendicular to the given straight line.

It will be seen that the Twelfth Proposition differs from the Eleventh only in this, that in the Eleventh the given point is *in* the given straight line and in the Twelfth the given point is *not in* the given straight line, but out of it—anywhere, in fact, *except* in it.

Note also that the words ‘of unlimited length’ are introduced into the enunciation of the Twelfth Proposition. The necessity of this is shown in fig. 47, where it is clear that if A B were limited (if we were not allowed to produce it), it would be impossible from the point C to draw a straight line perpendicular to A B.

Let A B, then (fig. 48), be a straight line of unlimited length, and let C be a given point without it.

It is required from the point C to draw a straight line perpendicular to A B.

As in the Eleventh Proposition so in the Twelfth, Euclid will require two points in the given straight line A B equally distant from the given point C.

To get them, he takes any point D on the other side of A B, and with the centre C and at the distance C D, by describing a circle E F G, he gets the two points (viz. F and G where the circle intersects the straight line A B) equally distant from C.

As a suggestive hint for the next step of the construction, let the learner ask himself whereabouts in F G he thinks the foot of the perpendicular will come. The answer is clearly, midway between F and G. This thought suggests the next step in the construction, which is:

Bisect F G[1] in the point H by the Tenth Proposition; upon joining C H Euclid says that C H will be perpendicular to A B.

It will be seen by reference to the Tenth Axiom, that all that is required in order to show that C H is at right angles (or, in other words, perpendicular) to A B, is to prove that the angle F H C is equal to the angle G H C.

With a view to the demonstration Euclid joins C F and C G.

It is to be hoped that most of the readers of this treatise will at once, without further help, write off the five steps of the demonstration by which it is proved that the angle F H C is equal to the angle G H C.

One hint, however, it is but fair to give, in order to put the learner on his guard against a mistake very commonly made in the demonstration of the Twelfth Proposition. It is often said the base C F is equal to the base C G, because they are sides of an equilateral triangle. But if the learner will look back to the construction, he will *not* find that C F G is to be made an equilateral triangle; but he *will* find that E F G is a circle, of which C is the centre, and therefore (as already often pointed out) he may be sure that in the demonstration he will have to say 'F C is equal to C G because they are radii of the same circle.' With this one precautionary hint, it is hoped that few will have any difficulty in writing out the five steps of the reasoning in the demonstration.[1]

---

[1] In the figure 48 (1), the construction required in order to get the point H is given in dotted lines. It is hoped that the learner by referring to the Tenth Proposition will understand the construction without further explanation.

In the hints accompanying the last four problems, the connection between the construction and demonstration is, some may think, to a needless degree pressed on the learner's notice. It is done purposely.

Beginners do not, generally, link together the construction and the demonstration. When they have learned to do so, their power of grappling with a proposition of Euclid is greatly and *most legitimately* increased.

It remains only to write out the Twelfth Proposition fully.

PROPOSITION XII. PROBLEM.

General Enunciation.

A straight line being given of unlimited length, and also a point without it:—

It is required from the given point to draw a straight line perpendicular to the given straight line.

Particular Enunciation.

Let A B (fig. 48(a)) be the given straight line, which may be produced to any length both ways; and let C be the given point without it:—

---

[1] If any learner still looks for guiding hints, he can take them from those given for the Eleventh Proposition.

It is required to draw from the point C a straight line perpendicular to A B.

Construction.

1. On the other side of A B take any point D.

2. From the centre C, at the distance C D, describe a circle E F G, cutting A B in the points F and G.

3. Bisect F G in the point H (Prop. 10).

4. Join C H.

Then will C H be perpendicular to A B.

5. For the demonstration, join C F and C G.

Demonstration.

1. The side F H is equal to the side H G, because it was made so.

2. And C H is common to both triangles.

3. Wherefore the two sides F H, H C of the triangle F H C are equal to the two sides G H, H C of the triangle G H C.

4. And the base C F is equal to the base C G, because they are radii of the same circle.

5. Therefore by the Eighth Proposition the included angle F H C is equal to the included angle G H C.

And these are adjacent angles; but when one straight line falling on another straight line makes the adjacent angles equal, each of them is a right angle, and the straight line which stands on the other is called a perpendicular to it (Def. 10).

Wherefore C H is perpendicular to A B, and it is drawn from the point C. Q.E.F.

The writer has now completed the task which he undertook, viz., to give a familiar explanation of the first twelve propositions of Euclid. His object all through has been to teach beginners how to learn Euclid. The aim of his explanations and observations has been throughout to prevent Euclid being, to the reader, merely a book of words, and to make it to him a book of sound and solid meaning.

He hopes that he will go through the propositions that follow successfully and with benefit to himself, by following the method of learning them here pointed out. Namely :—

First. By fixing in his mind what is given, and what has to be done or to be proved.

Secondly. By observing the construction by which Euclid professes to effect what he wants to effect; *building up* his figure according to Euclid's directions.

Thirdly. By following out the demonstration in order to discover if Euclid therein shows that the construction has effected what he *says* it has.

He would even recommend the learner to do more than this. He would advise him, in his further progress through the First Book, to carry on the method of reading Euclid suggested by the writer's treatment of the last four propositions.

When he has mastered the enunciation and construction, let him close his book and try whether he cannot make out for himself the proof that the construction has effected what was required.

If he cannot succeed without help, then let him note all the references, and observe the order in which they succeed each other in Euclid's demonstration. These references will afford suggestive hints analogous to those given in the preceding pages to encourage and help the learner to write out the demonstrations of the Ninth, Tenth, Eleventh, and Twelfth Propositions before reading over the demonstrations as given in Euclid.

When Euclid is thus read it becomes much more interesting. It has all the entertainment of solving riddles. Riddles, indeed, of a very high order are Euclid's problems and theorems.

This method has also the effect of fixing the propositions in the learner's mind far more permanently than if he merely reads through Euclid's demonstration, so as to follow his meaning, but without exerting independent thought.

Still, the learner must be cautioned while reading Euclid in this way not to hurry on too much, but to compare his demonstration with Euclid's, and to write it out in the best form with attention and care.

It is related in Sir David Brewster's 'Life of Sir Isaac Newton,' that having purchased a copy of Euclid, soon after entering at Trinity College, he examined the problems, and found the truths which they enunciated so self-evident, that he expressed his astonishment that any person should have taken the trouble of writing a demonstration of them. He therefore threw aside Euclid as a 'trifling book,' and set himself to the study of Descartes' Geometry. The neglect which he had shown of the elementary truths of geometry he afterwards regarded as a mistake in his mathematical studies, and he expressed to Dr. Pemberton 'his regret that he had applied himself to the works of Descartes and other algebraical writers before he had considered the elements of Euclid with that *attention* that so excellent a writer deserved.'—Sir Isaac Newton's words as given in Pemberton's view of his philosophy.

Although the writer's task is now done, he will, before parting from those who have accompanied him thus far, give them a few landmarks which they may find useful in onward journey.

In the Thirteenth Proposition Euclid introduces a new truth, which, combined with those going before, enables him through the eight following propositions to prove a variety of interesting truths about triangles.

The only proposition among them which perplexes beginners is the Thirteenth itself, which is as follows:

The angles which one straight line makes with another straight line on the same side of it (see fig. 49) are either two right angles, or together equal to two right angles.

If the adjacent angles are *equal* (see fig. 49(1)), each of them is a right angle (Def. 10), and therefore the two are two right angles.

If they are not equal, as in fig. 49(2), he proves that the two angles are together equal to two right angles, in the following manner:

From the point B he draws B E at right angles to C D (fig. 50(1)).

The nature of Euclid's proof will be seen better by splitting up his figure into three. Fig. 50(1) is Euclid's figure; fig. 50(2) is the pair of angles D B A, A B C, which will be found in Euclid's figure; and fig. 50(3) is the pair of angles, D B E, E B C, also found in Euclid's figure.

Of these we know that the pair D B E, E B C (fig. 50(3)) are together two right angles; and he proves that the pair D B A, A B C (fig. 50(2)) are equal to two right angles by showing that they are equal to the pair D B E, E B C, (fig. 50(3)). And this he does by showing that each pair (viz. D B A, A B C together, and D B E, E B C together) are equal to the three angles D B A, A B E, E B C together (fig. 50(1)).

Thus our old friend, the man that went by the train, here re-appears as the three angles D B A, A B E, E B C; while the pair D B A and A B C, and the pair D B E and E B C are respectively the London man and the Windsor man.

This figure and the above explanation may help the learner to understand the Thirteenth Proposition, which is sometimes found confusing from the number of angles named.

Having mastered this proposition, he will find the next eight propositions easy, each one leading on to the next.

After this, Euclid wants, in order to make a further advance, to be able to make an angle equal to a given angle: he shows how this may be done in the Twenty-second and Twenty-third Propositions, and makes use of this new power in the Twenty-fourth and Twenty-fifth.

Then follows the Twenty-sixth Proposition, in which Euclid proves the equality of triangles by a new set of conditions, in addition to those named in the Fourth and the Eighth Propositions.

These three propositions, the Fourth, the Eighth, and the Twenty-sixth, are linked together as being the three propositions in the First Book by which, under certain given conditions, it can be proved that two triangles are equal in every respect.

This closes the First Part of the First Book. After that Euclid passes to the consideration of parallel lines, by means of which he develops various, and some very surprising truths, to the end of the First Book.

And when the learner shall have gained an intelligent insight into the First Book of Euclid, he may be assured, for his comfort, that he will have made a most valuable acquisition, the benefit of which he will feel in every study he may enter upon, and through all his future life.

Postscriptum.—Before bidding his readers finally farewell, the writer would add one word more.

In teaching Euclid, orally, according to the foregoing method, he has had, these many years, great happiness, and he hopes some success. But so much in vivâ voce teaching depends on the voice and the eye, and the being able to read in the pupil's countenance whether he has taken in the explanation and conquered the difficulty, that he doubts—at least he is desirous to know, through the reader's experience, whether it is possible to transfer to print a method of teaching which has answered by word of mouth. Does he build too much on the good will of his readers if he asks those who have gone through the treatise kindly to let him know if they have found it helpful—if it has led them on to appreciate geometrical reasoning, and take an interest in Euclid—if it has cleared up the difficulties in their way, and enabled them to master the propositions of Euclid which follow readily and with satisfaction to themselves?

On the other hand, he would be glad to know if learners have been wearied with too profuse explanation and iteration. Though here he may repeat what was said at the beginning, that they who can swim without corks have only to throw their corks away.

In fact, any suggestions from teachers or learners that may make the book more useful, if it should reach a second edition, will be most thankfully received by the writer, and will be treated as friendly and confidential communications.

Church House, Windsor:
October, 1873.

EXPLANATION OF THE WOOD-CUT ON THE TITLE-PAGE.

1. The First Proposition is the first link of the chain.

2. The Second Proposition, depending on the First, is the second link.

3. The Third Proposition, depending directly on the Second, and through the Second on the First, is the next link.

4. These three propositions are represented by smaller links, they being propositions subsidiary to those of the main chain of propositions, which starts from the Fourth Proposition as from a fresh beginning.

5. Upon the Fourth hangs the Fifth Proposition; therefore the link (5) in the main chain is suspended from (4).

6. Upon the Fourth hangs also the Sixth Proposition; but no proposition among the first Twelve depends upon the Sixth, and so no link hangs from that numbered (6).

Obs. In the construction of the Fifth and Sixth Propositions it is required from the greater of two straight lines to cut off a part equal to the less (Prop. III.). Therefore a second and special link connects those numbered (5) and (6) with the one numbered (3).

7. The Seventh Proposition depends on the Fifth; therefore link (7) is hung on (5).

8. On the Seventh Proposition depends the Eighth; therefore link (8) is hung on (7).

9. The Eighth Proposition enables us to solve the Ninth; therefore link (9) hangs from (8).

10. For the construction of the Tenth Proposition the Ninth is required; therefore link (10) is hung from (9); but as it is demonstrated by the Fourth, a special link in the chain connects (4) and (10).

11. The Eleventh Proposition depends on the Eighth, therefore link (11) is hung from link (8); but as the construction requires that from the greater of two given straight lines a part be cut off equal to the less (Prop. III.), a special link connects those marked (11) and (3).

12. In the construction of the Twelfth, it is necessary to bisect a straight line; therefore, in the chain (12) is linked to (10). It is demonstrated by the Eighth, therefore a special link connects those marked (8) and (12).

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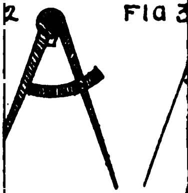


FIG 3

G 5

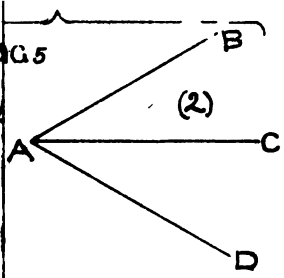
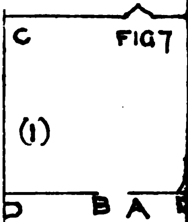


FIG 7



G 10

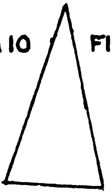


FIG 11

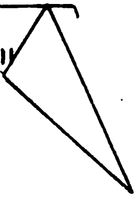
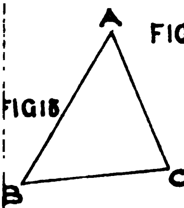


FIG 18



FIG

FIG 19

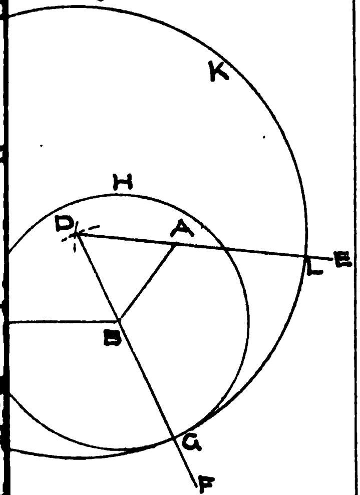


FIG 18